Darmon's points and quaternionic Shimura varieties

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Abstract. In this paper, we generalize a conjecture due to Darmon and Logan (see [DL03] and [Dar04], chapter 8) in an adelic setting. We study the relation between our construction and Kudla's works on cycles on orthogonal Shimura varieties. This relation allows us to conjecture a Gross-Kohnen-Zagier theorem for Darmon's points.

1. Introduction

The theory of complex multiplication gives a collection of *Heegner points* on elliptic curves over $\mathbf{Q}$, which are defined over class fields of imaginary quadratic fields. These points allowed to prove Birch and Swinnerton-Dyer's conjecture over $\mathbf{Q}$ for analytic rank 1 curves, thanks to the work of Gross-Zagier and Kolyvagin.

Let us briefly recall the construction of Heegner points. If E is an elliptic curve over $\mathbf{Q}$ then we know that E is modular. Let N be the conductor of E. There exists a modular form $f \in S_2(N)$ such that $L(E,s) = L(f,s)$. Denote by $\Phi_N : \Gamma_0(N)\backslash\mathcal{H} \longrightarrow E(\mathbf{C})$ the modular uniformization which is obtained by taking the composition of the map $z_0 \in \mathcal{H} \mapsto c\int_{i\infty}^{z_0} 2\pi i f(z)\mathrm{d}z$ (here c denotes the Manin constant) with the Weierstrass uniformization. Let $z_0 \in \mathcal{H} \cap K$, where $K/\mathbf{Q}$ is an imaginary quadratic field. A Heegner point is given essentially by $2\pi i \int_{i\infty}^{z_0} f(z)\mathrm{d}z$ modulo periods of f. It is the Abel-Jacobi image of z_0 in $\mathbf{C}/\Lambda_E \simeq E(\mathbf{C})$. The theory of complex multiplication shows that these points are defined over class fields of K.

In [Dar04], Darmon gives a conjectural construction of *Stark-Heegner points*, which is a generalization of classical Heegner points. These points should help us to understand, on one hand the Birch and Swinnerton-Dyer conjecture, on the other hand Hilbert's twelfth problem.

In more concrete terms, assume that F is a totally real number field of narrow class number 1. Let τ_j be its archimedean places, and K/F some quadratic "ATR" extension (i.e. K has exactly one complex place). Darmon defines a collection of points on elliptic curves E/F which are expected to be defined over class fields of K. In this case, the (conjectural, but partially proved by Skinner - Wiles) modularity of E gives the existence of a Hilbert modular form f on $\mathcal{H}^r$ whose periods appear as a tensor product of periods of $E_{\tau_j} = E \otimes_{F,\tau_j} \mathbf{C}$. The construction explained in [DL03] can be seen as an exotic Abel-Jacobi map.

In this paper, we generalize Darmon's contruction by removing the hypothesis "ATR" on K (but we assume that K is not CM) and the technical hypothesis that F has narrow class number 1. We replace the Hilbert modular variety used in the "ATR" case by a general quaternionic Shimura variety and define a suitable Abel-Jacobi map. We are able to specify the invariants of the quaternion algebra using local epsilon factors and to give a conjectural Gross-Zagier formula for these points. We conclude the paper by establishing a relation to Kudla's study of cycles on orthogonal Shimura varieties, in order to give a Gross-Kohnen-Zagier type conjecture.

Let us summarize the main construction of this paper. Let F be a totally real field of degree d and let $\tau_1, \dots, \tau_d$ be its archimedean places. Fix $r \in \{2, \dots, d\}$, and and a quadratic extension K/F such that the set of archimedean places of F that split completely in K is $\{\tau_2, \dots, \tau_r\}$. Let B/F be a quaternion algebra which splits at $\tau_1, \dots, \tau_r$ and ramifies at $\tau_{r+1}, \dots, \tau_d$. Let $G = \mathrm{Res}_{F/\mathbf{Q}} B^\times$. We will denote by $\mathrm{Sh}_H(G,X)$ the quaternionic Shimura variety of level H (a compact open subgroup of $G(\mathbf{A}_f)$) whose complex points are given by

$$\mathrm{Sh}_H(G,X)(\mathbf{C}) = G(\mathbf{Q})\backslash(\mathbf{C} \smallsetminus \mathbf{R})^r \times G(\mathbf{A}_f)/H.$$

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Fix an embedding $q : K \hookrightarrow B$. There is an action of $(K \otimes \mathbf{R})_+^\times / (F \otimes \mathbf{R})^\times$ on $(\mathbf{C} \setminus \mathbf{R})^r$. By considering a suitable orbit of this action, we obtain a real cycle T_b of dimension $r-1$ on $\mathrm{Sh}_H(G,X)(\mathbf{C})$. Using the theorem of Matsushima and Shimura, we deduce that there exists an r-cycle Δ_b on $\mathrm{Sh}_H(G,X)(\mathbf{C})$ such that $\partial\Delta_b$ is an integral multiple of $\mathscr{T}_b$.

Let E/F be an elliptic curve, assumed modular, i.e., there exists a Hilbert modular eigenform $\tilde{\varphi}$ satisfying $L(E,s) = L(\tilde{\varphi}, s)$. We will assume that this form corresponds to an automorphic form φ on B by the Jacquet-Langlands correspondence. There exists a holomorphic differential form ω_φ of degree r on $\mathrm{Sh}_H(G,X)(\mathbf{C})$ naturally attached to φ. In general, the set of periods of ω_φ is a dense subset of $\mathbf{C}$. Fix some character β of the set of connected components of $(K \otimes \mathbf{R})_+^\times/(F \otimes \mathbf{R})^\times$. Following Darmon we define a modified differential form ω_φ^β whose periods are, assuming Yoshida's period conjecture, a lattice, homothetic to some sublattice of the Neron lattice of E.

The image of (a suitable multiple of) the complex number $\int_{\Delta_b} \omega_\varphi^\beta$ in $\mathbf{C}/\Lambda_E$ is independent of the choice of Δ_b. Hence it defines by Weierstrass uniformization a point P_b^β in $E(\mathbf{C})$. We conjecture

Conjecture (5.1.1)**.** *$P_b^\beta = \Phi\left(\int_{\Delta_b} \omega_\varphi^\beta\right) \in E(\mathbf{C})$ lies in $E(K^{ab})$ and*

$$\forall a \in \mathbf{A}_K^\times \qquad \mathrm{rec}_K(a) P_b^\beta = \beta(a_\infty) P^\beta_{q_{\mathbf{A}}(a)b}.$$

Let us assume this conjecture is true and denote by K_b^+ the field of definition of P_b^β. Let $\pi = \pi(\varphi)$ be the automorphic representation generated by φ; fix a character $\chi : \mathrm{Gal}(K_b^+/K) \to \mathbf{C}^\times$. Denote by $\varepsilon(\pi \times \chi, \frac{1}{2})$ the sign in the functional equation of the Rankin-Selberg L-function $L(\pi \times \chi, s)$ and by $\eta_K : F_{\mathbf{A}}^\times / F^\times \mathrm{N}_{K/F}(K_{\mathbf{A}}^\times) \to \{\pm 1\}$ the quadratic character of K/F. The following proposition proves that B is uniquely determined by K and the isogeny class of E/F.

Proposition (5.3.1)**.** *Let $b \in \widehat{B}^\times$ and assume conjecture 5.1.1. If*

$$e_{\overline{\chi}}(P_b^\beta) = \sum_{\sigma \in \mathrm{Gal}(K_b^+/K)} \chi(\sigma) \otimes P_b^\beta \in E(K_b^+) \otimes \mathbf{Z}[\chi]$$

is not torsion, then :

$$\forall v \nmid \infty \qquad \eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \mathrm{inv}_v(B_v) \quad \text{and} \quad \varepsilon(\pi \times \chi, \frac{1}{2}) = -1.$$

The last part of this paper is focused on a conjecture in the spirit of the Gross-Kohnen-Zagier theorem. Assume that $E(F)$ has rank 1. Denote by P_0 some generator modulo torsion. For each totally positive $t \in O_F$ such that (t) is square free and prime to $d_{K/F}$, denote by $K[t]$ the quadratic extension $K[t] = F(\sqrt{-D_0 t})$, where $D_0 \in F$ satisfies $\tau_j(D_0) > 0$ if and only if $j \in \{1, r+1, \ldots, d\}$. Let $P_{t,1}$ be Darmon's point obtained for $K[t]$ and $b = 1$, and set

$$P_t = \mathrm{Tr}_{K[t]_1^+/F} P_{t,1}.$$

The point P_t is in $E(F)$ and there exists some integer $[P_t] \in \mathbf{Z}$ such that $P_t = [P_t]P_0$. In the spirit of conjecture 5.3 of [DT08] we conjecture that :

Conjecture (6.3.5)**.** *There exists a Hilbert modular form g of level 3/2 such that the $[P_t]$s are proportional to some Fourier coefficients of g.*

In our attempt to adapt Yuan, Zhang and Zhang's proof in the CM case [YZZ09] to prove this conjecture, we obtained a relation between Darmon's points and Kudla's program, see Proposition 5.5.3.2.

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2. Quaternionic Shimura varieties

In this section we recall some properties of Shimura varieties associated to quaternion algebras. The standard references are Reimann's book [Rei97] and [Mil05]. The content of this section is more or less the transcription to Shimura varieties of what is done for curves in [CV07] and [Nek07].

Let F be a totally real field of degree $d = [F : \mathbf{Q}]$ and $\tau_1, \ldots, \tau_d$ its archimedean places. Denote by $\overline{\mathbf{Q}} \subset \mathbf{C}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ so $\tau_j : F \hookrightarrow \overline{\mathbf{Q}}$. Fix $r \in \{2, \ldots, d\}$ and a finite set S_B of non-archimedean primes satisfying

$$|S_B| \equiv d - r \bmod 2.$$

Let B be the unique quaternion algebra over F ramified at the set

$$\mathrm{Ram}(B) = \{\tau_{r+1}, \ldots, \tau_d\} \cup S_B.$$

For each $j \in \{1, \ldots, d\}$ we put $B_{\tau_j} = B \otimes_{F,\tau_j} \mathbf{R}$. It is not necessary but more convenient to fix for each $j \in \{\tau_1, \ldots, \tau_r\}$ an $\mathbf{R}$-algebra isomorphism

$$B_{\tau_j} \xrightarrow{\sim} M_2(\mathbf{R}).$$

The constructions given in this paper are independent on the choice of these isomorphisms, as in the author's PhD thesis [Gär11].

Let G be the algebraic group over $\mathbf{Q}$ satisfying $G(A) = (B \otimes_{\mathbf{Q}} A)^\times$ for every commutative $\mathbf{Q}$-algebra A. We will denote by $\mathrm{nr} : G(A) \longrightarrow (F \otimes_{\mathbf{Q}} A)^\times$ the reduced norm and by Z the center of G. For $j \in \{1, \ldots, d\}$ let G_j be the algebraic group over $\mathbf{R}$ given by $G_j = G \otimes_{F,\tau_j} \mathbf{R}$; thus $G_{\mathbf{R}}$ decomposes as $G_1 \times \cdots \times G_d$. For any abelian group A, denote by $\widehat{A}$ the group $A \otimes \widehat{\mathbf{Z}}$.

Let X be the $G(\mathbf{R})$-conjugacy class of the morphism $h : \mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m,\mathbf{C}}) \longrightarrow G(\mathbf{R}) = G_1(\mathbf{R}) \times \cdots \times G_d(\mathbf{R})$ defined by

$$x + iy \longmapsto \left(\underbrace{\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \ldots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}}_{r \text{ times}}, \underbrace{1, \ldots, 1}_{d-r \text{ times}} \right).$$

The set X has a natural complex structure [Mil90] and the following map is an holomorphic isomorphism between X and $(\mathbf{C} \smallsetminus \mathbf{R})^r$:

$$ghg^{-1} \longmapsto g \cdot (i, \ldots, i) = \left(\frac{a_1 i + b_1}{c_1 i + d_1}, \ldots, \frac{a_r i + b_r}{c_r i + d_r} \right),$$

where $g = (g_1, \ldots, g_d) \in G(\mathbf{R})$ and for $j \in \{1, \ldots, r\}$ g_j is identified with $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$.

Quaternionic Shimura varieties. Let H be an open-compact subgroup of $\widehat{B}^\times$. The quaternionic Shimura varieties considered in this paper are algebraic varieties $\mathrm{Sh}_H(G, X)$ whose complex points are given by

$$\mathrm{Sh}_H(G, X)(\mathbf{C}) = B^\times \backslash (X \times \widehat{B}^\times / H)),$$

where the left-action of $B^\times$ and the right-action of H are given by

$$\forall k \in B^\times \ \forall h \in H \ \forall (x, b) \in X \times \widehat{B}^\times \qquad k \cdot (x, b) \cdot h = (kx, kbh).$$

Such Shimura varieties are defined over some number field called the reflex field. In our case this number field is

$$F' = \mathbf{Q} \left(\sum_{j=1}^{r} \tau_j(\alpha), \ \alpha \in F \right) \subset \overline{\mathbf{Q}} \subset \mathbf{C}.$$

We will denote by $[x, b]_H$ the element of $\mathrm{Sh}_H(G, X)(\mathbf{C})$ represented by (x, b) and by $[x, b]_{H\widehat{F}^\times}$ the corresponding element of the modified variety $\mathrm{Sh}_H(G/Z, X)(\mathbf{C}) = B^\times \backslash (X \times \widehat{B}^\times / HZ))$.

Remark 2.1.1. The complex Shimura varieties are compact whenever $B \neq M_2(F)$. The Hilbert modular varieties used by Darmon in [Dar04] chapter 7 and 8 are obtained when $B = M_2(F)$ and $r = d$.

The Shimura varieties form a projective system $\{\mathrm{Sh}_H(G, X)\}_H$ indexed by open compact subgroups in $\widehat{B}^\times$. The transition maps $\mathrm{pr} : \mathrm{Sh}_H(G, X) \to \mathrm{Sh}_{H'}(G, X)$ are defined on complex points by

$$[x, b]_H \to [x, b]_{H'}.$$

There is an action of $\widehat{B}^\times$ on the projective system $\{\mathrm{Sh}_H(G,X)\}_H$. The right multiplication by $g\in\widehat{B}^\times$ induces an isomorphism $[\cdot g] : \{\mathrm{Sh}_H(G,X)\}_H \xrightarrow{\sim} \{\mathrm{Sh}_H(G,X)\}_{g^{-1}Hg}$, defined on complex points by

$$[\cdot g][x,b]_H = [x,bg]_{g^{-1}Hg}.$$

Complex conjugation. Fix $j\in\{1,\dots,r\}$. Let $h_j : \mathbf{S}\to G_{j,\mathbf{R}}$ be the morphism obtained by composing h with the j-th projection $G_{\mathbf{R}}\to G_{j,\mathbf{R}}$ and X_j the $G_j(\mathbf{R})$-conjugacy class of h_j. For $x_j = g_jh_jg_j^{-1}\in X_j$, the set $\mathrm{Im}(g_jh_jg_j^{-1})$ is a maximal anisotropic $\mathbf{R}$-torus in $G_{j,\mathbf{R}}$. The map $\ell_j : x_j\mapsto \mathrm{Im}(x_j)$ satisfies $|\ell_j^{-1}(\ell_j(x_j))| = 2$, thus there exists a unique antiholomorphic and $G_{j,\mathbf{R}}$-equivariant involution

$$t_j : X_j \longrightarrow X_j$$

such that

$$\forall x_j\in X_j \qquad \ell_j^{-1}(\ell_j(x_j)) = \{x_j, t_j(x_j)\}.$$

More precisely, under the identification $X_j \xrightarrow{\sim} \mathbf{C}\smallsetminus\mathbf{R}$, the map ℓ_j satisfies $\ell_j(x+iy) = \left\{\begin{pmatrix} x & y\\ -y & x\end{pmatrix}\right\}$ and $\ell_j^{-1}(\ell_j(x+iy)) = \{x+iy, x-iy\}$. Note that the map t_j can be extended to complex points of the Shimura varieties by $t_j([x,b]_H) = [t_j(x),b]_H$; t_j acts trivially on X_k for $k\neq j$.

Differential forms. In this section we recall some facts concerning differential forms on Shimura varieties. We will denote by $\Omega_H = \Omega_{H/F'}$ the sheaf of differentials of degree r on $\mathrm{Sh}_H(G,X)$ and by Ω_H^{an} the sheaf of holomorphic r-differentials on $\mathrm{Sh}_H(G,X)(\mathbf{C})$, provided that $\mathrm{Sh}_H(G,X)$ is smooth. Recall that the GAGA principle gives us the following isomorphism between global sections

$$\Gamma(\mathrm{Sh}_H(G,X),\Omega_H)\otimes_{F'}\mathbf{C} \xrightarrow{\sim} \Gamma(\mathrm{Sh}_H(G,X)(\mathbf{C}),\Omega_H^{\mathrm{an}}).$$

Notice that in general, $\mathrm{Sh}_H(G,X)$ is not smooth. In this last case we will fix some integer $n\geq 3$ such that for each $\mathfrak{p}$ in $\mathrm{Ram}(B)$ we have $\mathfrak{p}\nmid n\mathcal{O}_F$ and for each $v\mid n\mathcal{O}_F$ isomorphisms $\iota_v : B_v \xrightarrow{\sim} M_2(F_v)$. The group

$$H' = \left\{(h_v)\in H,\ \text{s.t. } \forall v\mid n\mathcal{O}_F\ h_v\equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \mod n\mathcal{O}_{F_v}\right\}$$

is of finite index in H and $\mathrm{Sh}_{H'}(G,X)$ is smooth. The map $\mathrm{Sh}_{H'}(G,X)\to\mathrm{Sh}_H(G,X)$ is a finite covering. We define $\Omega_H = \frac{1}{[H:H']}\sum_{\sigma\in H/H'}\sigma\Omega_{H'} = (\Omega_{H'})^H$. By abuse of language, we shall call an element of $\Gamma(\Omega_H) = \Gamma(\mathrm{Sh}_H(G,X),\Omega_H) = (\sum_{\sigma\in H/H'}\sigma)\Gamma(\mathrm{Sh}_{H'}(G,X),\Omega_{H'})$ a global r-form on $\mathrm{Sh}_H(G,X)$. Remark that the space of global holomorphic r-forms $\varinjlim_H \Gamma(\Omega_H^{\mathrm{an}})$ is equipped with a canonical action of $\widehat{B}^\times$ given by pull-backs $[\cdot g]^*$.

Let $\varepsilon\in\{\pm1\}^r$ and denote by $\Gamma((\Omega_H^{\mathrm{an}})^\varepsilon)$ the space of r-forms on $\mathrm{Sh}_H(G,X)(\mathbf{C})$ which are holomorphic (resp. anti-holomorphic) in z_j if $\varepsilon_j=+1$ (resp. if $\varepsilon_j=-1$). The maps t_j pulled-back on $\Gamma((\Omega_H^{\mathrm{an}})^\varepsilon)$ satisfy

$$t_j^* : \Gamma((\Omega_H^{\mathrm{an}})^\varepsilon) \longrightarrow \Gamma((\Omega_H^{\mathrm{an}})^{\varepsilon'})$$

where $\varepsilon'_k = \varepsilon_k$ for $k\neq j$ and $\varepsilon'_j = -\varepsilon_j$.

When $\sigma\in\prod_{j=2}^r\{\pm1\}$ we will define $e_j\in\{0,1\}$ by $\sigma_j = (-1)^{e_j}$ and t_σ^* by $\prod_{j=2}^r (t_j^*)^{e_j}$. Let $\beta : \prod_{j=2}^r\{\pm1\}\to\{\pm1\}$ be a character and $\omega\in\Gamma(\Omega_H^{\mathrm{an}})$. We shall denote by ω^β the element $\omega^\beta = \sum_{\sigma\in\{\pm1\}^{r-1}}\beta(\sigma)t_\sigma^*(\omega)$ of $\bigoplus_\varepsilon\Gamma((\Omega_H^{\mathrm{an}})^\varepsilon)$.

Automorphic forms. Let S_2^H be the space $S^H_{2,\dots,2,0,\dots,0}(B_{\mathbf{A}}^\times)$ of functions $\varphi : B_{\mathbf{A}}^\times \simeq G(\mathbf{R})\times\widehat{B}^\times \longrightarrow \mathbf{C}$ satisfying the following properties :

(1) $\forall g\in B^\times\ \forall b\in B_{\mathbf{A}}^\times \qquad \varphi(gb)=\varphi(b)$,
(2) $\forall g\in(\mathbf{R}^\times)^r\times G_{r+1}(\mathbf{R})\times\cdots\times G_d(\mathbf{R})\subset G(\mathbf{R})\ \forall b\in B_{\mathbf{A}}^\times \qquad \varphi(bg)=\varphi(b)$,
(3) $\forall h\in H\ \forall b\in B_{\mathbf{A}}^\times \qquad \varphi(bh)=\varphi(b)$,
(4) $\forall g\in B_{\mathbf{A}}^\times\ \forall(\theta_1,\dots,\theta_r)\in\mathbf{R}^r$

$$\varphi\left(g\left[\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix},\dots,\begin{pmatrix}\cos\theta_r & -\sin\theta_r\\ \sin\theta_r & \cos\theta_r\end{pmatrix},1,\dots,1\right]\right) = e^{-2i\theta_1}\times\cdots\times e^{-2i\theta_r}\varphi(g),$$

(5) For all $g \in B_{\mathbf{A}}^{\times}$, the map

$$(x_1+iy_1,\dots,x_r+iy_r) \mapsto \frac{1}{y_1\dots y_r}\varphi\left(g\left[\begin{pmatrix} y_1 & x_1 \\ 0 & 1\end{pmatrix},\dots,\begin{pmatrix} y_r & x_r \\ 0 & 1\end{pmatrix},1,\dots,1\right]\right)$$

is holomorphic on $\mathcal{H}^r$ where $\mathcal{H}$ denotes the Poincaré upper-half plane.

Remark that we do not need any assumption to obtain cuspidal forms as B will be assumed to differ from $M_2(F)$.

There is an action of $\widehat{B}^{\times}$ on $S_2 = \bigcup_H S_2^H$ defined by

$$\forall g \in \widehat{B}^{\times},\ \forall \varphi \in S_2,\ \forall x \in B_{\mathbf{A}}^{\times} \qquad g\cdot\varphi(x) = \varphi(xg);$$

thus S_2^H is the space of H-invariant functions in S_2.

By modifying the properties 4 and 5 above we obtain the following new definition :

Definition 2.1.2. Let $\varepsilon : \{\tau_1,\dots,\tau_r\} \longrightarrow \{\pm 1\}$ and $\varepsilon_i = \varepsilon(\tau_i)$. The space $(S_2^{\varepsilon})^H$ is the space of maps $\varphi : B_{\mathbf{A}}^{\times} \simeq G(\mathbf{R}) \times \widehat{B}^{\times} \longrightarrow \mathbf{C}$ satisfying 1-3 above and

4'. for all $g \in B_{\mathbf{A}}^{\times}$ and $(\theta_1\dots\theta_r) \in \mathbf{R}^r$

$$\varphi\left(g\left(\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1\end{pmatrix},\dots,\begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r\end{pmatrix},1,\dots,1\right)\right) = e^{-2i\varepsilon_1\theta_1}\times\dots\times e^{-2i\varepsilon_r\theta_r}\varphi(g)$$

5'. for all $g \in B_{\mathbf{A}}^{\times}$ the map

$$(x_1+iy_1,\dots,x_r+iy_r) \mapsto \frac{1}{y_1\dots y_r}\varphi\left(g\left(\begin{pmatrix} y_1 & x_1 \\ 0 & 1\end{pmatrix},\dots,\begin{pmatrix} y_r & x_r \\ 0 & 1\end{pmatrix},1,\dots,1\right)\right)$$

is holomorphic (resp. anti-holomorphic) in $z_j = x_j + iy_j \in \mathcal{H}$ if $\varepsilon_j = 1$ (resp. $\varepsilon_j = -1$).

We will denote by $S_2^{\hat{F}^{\times}}$ (resp. $(S_2^{\varepsilon})^{\hat{F}^{\times}}$) the space of elements in S_2 (resp. S_2^{ε}) which are $\hat{F}^{\times}$-invariant.

We are now able to affirm the existence of relations between r-forms on $\mathrm{Sh}_H(G,X)(\mathbf{C})$ and automorphic forms :

Proposition 2.1.3. *There exist bijections compatible with the $\widehat{B}^{\times}$-action between the following spaces :*

$\Gamma(\Omega_H^{\mathrm{an}})$	*and*	S_2^H
$\Gamma((\Omega_H^{\mathrm{an}})^{\varepsilon})$	*and*	$(S_2^{\varepsilon})^H$
$\Gamma(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),(\Omega_H^{\mathrm{an}})^{\varepsilon})$	*and*	$(S_2^{\varepsilon})^{H\widehat{F}^{\times}}$

This statement is completely analogous to section 3.6 of [CV07], see [Gär11], Propositions 1.2.2.4 and 1.2.2.5 for more details.

Matsushima-Shimura theorem. The decomposition of the cohomology of quaternionic Shimura varieties given by Matsushima-Shimura theorem will be usefull in the following sections. Let us recall this result when $B \neq M_2(F)$ [MS63] and [Fre90]. Denote by h_F^+ the narrow class number of F.

Theorem 2.1.4. *Let $m \in \{0,\dots,2r\}$. We have the following decomposition :*

$$H^m(\mathrm{Sh}_H(G,X)(\mathbf{C}),\mathbf{C}) \simeq \begin{cases} \left(\mathrm{Vect}\bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\|a|=m/2}} \frac{\mathrm{d}z_i\wedge\mathrm{d}\overline{z_i}}{y_i^2}\right)^s & \text{if } m\neq r \\ \left(\mathrm{Vect}\bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\|a|=m/2}} \frac{\mathrm{d}z_i\wedge\mathrm{d}\overline{z_i}}{y_i^2}\right)^s \oplus \bigoplus_{\varepsilon\in\{\pm1\}^r}(S_2^{\varepsilon})^H & \text{if } m = r\end{cases}$$

and

$$H^m(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C}) \simeq \begin{cases} \left(\mathrm{Vect}\bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\|a|=m/2}} \frac{\mathrm{d}z_i\wedge\mathrm{d}\overline{z_i}}{y_i^2}\right)^{s'} & \text{if } m\neq r \\ \left(\mathrm{Vect}\bigwedge_{\substack{i\in a\subset\{1,\dots,r-1\}\\|a|=m/2}} \frac{\mathrm{d}z_i\wedge\mathrm{d}\overline{z_i}}{y_i^2}\right)^{s'} \oplus \bigoplus_{\varepsilon\in\{\pm1\}^r}(S_2^{\varepsilon})^{H\hat{F}^{\times}} & \text{if } m = r,\end{cases}$$

where s (resp. s') is the number of connected components of $\mathrm{Sh}_H(G,X)(\mathbf{C})$ *(resp. of* $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$*).*

3. Periods

3.1. **Yoshida's conjecture.** Let E/F be an elliptic curve, assumed modular in the sense that there exists a cuspidal, parallel weight two Hilbert modular form $\tilde{\varphi} \in S_2(\mathrm{GL}_2(F_{\mathbf{A}}))$ satisfying $L(E,s) = L(\tilde{\varphi}, s)$. We shall assume that the automorphic representation generated by $\tilde{\varphi}$ is obtained by the Jacquet-Langlands correspondence from $\varphi \in S_2^{H\widehat{F}^\times}(B_{\mathbf{A}}^\times)$.

Denote by $\pi = \pi_\infty \otimes \pi_f$ the automorphic representation of $B_{\mathbf{A}}^\times / F_{\mathbf{A}}^\times$ generated by φ. We shall assume until section 3.3, only for simplicity, that $\dim \pi_f^H = 1$.

Let $M = h^1(E)$ be the motive over F with coefficients in $\mathbf{Q}$ associated to E. Yoshida [Yos94] conjectures the existence of a rank 2^r motive M' over the reflex field F', with coefficients in $\mathbf{Q}$, satisfying $M' = \bigotimes_{\{\tau_1,\dots,\tau_r\}} \mathrm{Res}_{F/F'} M$. This motivic conjecture is the following :

Conjecture 3.1.1 (Yoshida, [Yos94])**.** *The motive M' over F' is isomorphic to the motive associated to the part $H^*(\mathrm{Sh}_{H\widehat{F}^\times}(G,X))^{(E)}$ of the cohomology for which Hecke eigenvalues are the same as E.*

While looking at the ℓ-adic realization, this conjecture is in fact the Langlands cohomological conjecture. This case is known, up to semi-simplification, thanks to Brylinski and Labesse in the case $B = \mathrm{M}_2(F)$ [BL84], Langlands in the case $B \neq \mathrm{M}_2(F)$ for primes of good reduction, [Lan79] and Reimann (- Zink) [Rei97, RZ91] for a more general cases.

Recall the following decompositions given by Yoshida in [Yos94] section 5.1, when we focus on $\tau' : F' \hookrightarrow \mathbf{C}$ induced by $\tilde{\tau'} : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$.
Betti cohomology. There exists an isomorphism of $\mathbf{Q}$-vector spaces

$$\mathscr{I} : M'_{\mathrm{B}} \xrightarrow{\sim} \bigotimes_{j=1}^{r} M_{B,\tau_j}$$

de Rham cohomology. The map

$$\mathscr{J} : M'_{\mathrm{dR}} \xrightarrow{\sim} \left(\bigotimes_{j=1}^{r} \left(M_{\mathrm{dR}} \otimes_{F,\tau_j} \overline{\mathbf{Q}} \right) \right)^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')}$$

is an isomorphism of F'-vector-spaces. The right hand side is a tensor product of $\overline{\mathbf{Q}}$-vector spaces and the action of $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/F')$ is given by $\bigotimes_{s\in\{\tau_1,\dots,\tau_r\}} (x_s \otimes_{F,s} a_s) \mapsto \bigotimes_{s\in\{\tau_1,\dots,\tau_r\}} (x_s \otimes_{F,\sigma s} \sigma(a_s))$.
Comparison isomorphisms. Let $I = \bigotimes_{j=1}^{r} I_{\tau_j}$, where

$$I_{\tau_j} : M_{\mathrm{B},\tau_j} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{F,\tau_j} \mathbf{C}$$

are isomorphisms of $\mathbf{C}$-vector spaces, and I' be the following isomorphism over $\mathbf{C}$:

$$I' : M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}.$$

The maps $I \circ (\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}})$ and $(\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}}) \circ I'$ satisfy :

$$(\star) \qquad I \circ (\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) = (\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}}) \circ I' : M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \bigotimes_{j=1}^{r} \left(M_{\mathrm{dR}} \otimes_{F,\tau_j} \mathbf{C} \right).$$

Yoshida's period conjecture consists of the isomorphisms $\mathscr{I}$, $\mathscr{J}$, I and I' satisfying $(\star)$. It is the Hodge-de Rham realization of the motivic conjecture above.
Complex conjugation : Let c_{τ_j} be the complex conjugation on M_{B,τ_j}. We will need the following hypothesis, which allows us to compare c_{τ_j} with t_j^* on $M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}$.

Hypothesis 3.1.2. *The action of t_j^* on $M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}$ corresponds via the isomorphism*

$$(\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) \circ (I')^{-1} : M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C} \longrightarrow M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \longrightarrow \left(\bigotimes_{k=1}^{r} M_{\mathrm{B},\tau_k} \right) \otimes_{\mathbf{Q}} \mathbf{C},$$

to the action of c_{τ_j} on M_{B,τ_j}.

3.2. **Lattices and periods.** Fix some $\omega_\varphi \neq 0$ in $F^r M'_{\mathrm{dR}}$. By definition of M', there exists a finite set of places S of F such that for $v \notin S$, $T_v \omega_\varphi = a_v(E)\omega_\varphi$.

Let $\Omega_{E/F}$ be the sheaf of differentials on E/F. Fix $\eta \neq 0 \in H^0(E, \Omega_{E/F}) = F^1 M_{dR}$. For $j \in \{1, \ldots, n\}$, let

$$\eta_j = \eta \otimes_{F,\tau_j} 1 \in H^0\left(E \otimes_{F,\tau_j} \overline{\mathbf{Q}}, \Omega_{(E\otimes_{F,\tau_j}\overline{\mathbf{Q}})/\overline{\mathbf{Q}}}\right) = \left(F^1 M_{\mathrm{dR}}\right) \otimes_{F,\tau_j} \overline{\mathbf{Q}}.$$

Then

$$\bigotimes_{j=1}^r \eta_j \in \left(\bigotimes_{j=1}^r \left(F^1 M_{\mathrm{dR}} \otimes_{F,\tau_j} \overline{\mathbf{Q}}\right)\right)^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')} = \mathscr{J}\left(F^r M'_{\mathrm{dR}}\right)$$

and there exists $\alpha \in F'^{\times}$ such that

$$\mathscr{J}(\alpha\omega_\varphi) = \eta_1 \otimes \cdots \otimes \eta_r.$$

Let $j \in \{1, \ldots, r\}$ and $E_j = E \otimes_{F,\tau_j} \mathbf{C}$. We shall denote by $H_1(E_j, \mathbf{Z})^{\pm}$ the eigenspaces of the complex conjugation action on $H_1(E_j, \mathbf{Z})$. Then

$$\left\{\int_\Upsilon \eta_j,\ \Upsilon \in H_1(E_j, \mathbf{Z})^{\pm}\right\} = \mathbf{Z}\Omega_j^{\pm},$$

where $\Omega_j^+ \in \mathbf{R} \smallsetminus \{0\}$ and $\Omega_j^- \in i\mathbf{R} \smallsetminus \{0\}$ are determined up to a sign. We fix the signs by imposing, e.g., $\mathrm{Re}\left(\Omega_j^+\right) > 0$ and $\mathrm{Im}\left(\Omega_j^-\right) > 0$.

Fix a character $\beta : \{1\} \times \prod_{j=2}^r \{\pm 1\} \to \{\pm 1\}$, and write $\beta = \prod_{j=2}^r \beta_j$. We set

$$\omega_\varphi^\beta = \left(\sum_{\sigma\in\{1\}\times\prod_{j=2}^r\{\pm1\}} \beta(\sigma) t_\sigma^*\right)\omega_\varphi = \prod_{j=2}^r \left(1 + \beta_j(-1)t_j^*\right)\omega_\varphi$$

and

$$\Omega^\beta = \prod_{j=2}^r \Omega_j^{\beta_j(-1)}.$$

The following identities

$$\left(\bigotimes_{j=1}^r M_{\mathrm{B},\tau_j}\right) \otimes_{\mathbf{Q}} \mathbf{C} = \bigotimes_{j=1}^r \mathrm{Hom}_{\mathbf{Z}}(H_1(E_j, \mathbf{Z}), \mathbf{C}) = \mathrm{Hom}_{\mathbf{Z}}\left(\bigotimes_{j=1}^r H_1(E_j, \mathbf{Z}), \mathbf{C}\right)$$

and Yoshida's conjecture show that the image of $\alpha\omega_\varphi^\beta$ under the map

$$(\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) \circ I'^{-1} = I^{-1} \circ (\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}}) : M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C} \longrightarrow \left(\bigotimes_{j=1}^r M_{B,\tau_j}\right) \otimes_{\mathbf{Q}} \mathbf{C}$$

is identified with the linear form

$$\left\{\begin{array}{lll} \bigotimes_{j=1}^r H_1(E_j, \mathbf{Z}) & \longrightarrow & \mathbf{C} \\ \Upsilon_1 \otimes \cdots \otimes \Upsilon_r & \longmapsto & \int_{\Upsilon_1\otimes\cdots\otimes\Upsilon_r} \bigotimes_{j=1}^r \left(1 + \beta_j(-1)t_j^*\right)\eta_j \end{array}\right. \tag{1}$$

Hypothesis 3.1.2 allows us to be more explicit. Let $\Upsilon_1 \otimes \cdots \otimes \Upsilon_r \in \bigotimes_{j=1}^r H_1(E_j, \mathbf{Z})$, then

$$\begin{aligned} \int_{\Upsilon_1\otimes\cdots\otimes\Upsilon_r} \bigotimes_{j=1}^r \left(1 + \beta_j(-1)t_j^*\right)\eta_j &= \left(\int_{\Upsilon_1} \eta_1\right) \prod_{j=2}^r \int_{\Upsilon_j} (1 + \beta_j(-1)t_j^*)\eta_j \\ &= \left(\int_{\Upsilon_1} \eta_1\right) \prod_{j=2}^r \int_{\Upsilon_j + \beta_j(-1)c_j\Upsilon_j} \eta_j. \end{aligned}$$

and the linear form (1) takes values in $\Lambda_1\Omega^\beta = (\mathbf{Z}\Omega_1^+ + \mathbf{Z}\Omega_1^-)\Omega^\beta$.

Under the dual isomorphism $\mathscr{I}^*$ of $\mathscr{I}$, the lattices

$$\bigotimes_{j=1}^r {}_{\mathbf{Z}} H_1(E_j, \mathbf{Z}) \subset \bigotimes_{j=1}^r {}_{\mathbf{Q}} M_{B,\tau_j}^* \quad \text{and} \quad \mathrm{Im}\left(H_r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{Z}) \longrightarrow (M'_B)^*\right)$$

are commensurable. Thus there exists $\xi \in \mathbf{Z} \smallsetminus \{0\}$ such that

$$\xi \mathrm{Im}\left(H_r(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z}) \longrightarrow (M'_B)^*\right) \subset \mathscr{I}^* \left(\bigotimes_{j=1}^{r}{}_{\mathbf{Z}} H_1(E_j,\mathbf{Z})\right).$$

This proves the following proposition :

Proposition 3.2.1. *Under the hypothesis made in this section (E is modular, the multiplicity one in Yoshida's motivic conjecture and 3.1.2), there exist $\alpha \in F'^{\times}$ and $\xi \in \mathbf{Z} \smallsetminus \{0\}$ such that*

$$\forall \gamma \in H_r(Sh_H(G,X)(\mathbf{C}),\mathbf{Z}),\ \forall \beta : \prod_{j=2}^{r}\{\pm 1\} \to \{\pm 1\}, \qquad \xi \int_\gamma \alpha\omega_\varphi^\beta \in \Lambda_1 \Omega^\beta.$$

3.3. **General case.** When $m_H(\pi) = \dim\, \pi_f^H(\varphi) > 1$ Yoshida's conjecture is the following

Conjecture 3.3.1. *The motive $H^r(\mathrm{Sh}_H(G,X))^{(E)}$ is isomorphic to* $\left(\bigotimes_{\{\tau_1,\dots,\tau_r\}} \mathrm{Res}_{F/F'} M\right)^{m_H(\pi)}$.

In general the motive $H^r(\mathrm{Sh}_H(G,X))^{(E)}$ has rank $\neq 2^r$. We shall provide Betti and de Rham realizations of a submotive $M' \subset H^r(\mathrm{Sh}_H(G,X))^{(E)}$ of rank 2^r and an isomorphism $M' \xrightarrow{\sim} \bigotimes_{\{\tau_1,\dots,\tau_r\}} \mathrm{Res}_{F/F'} M$.

We need $0 \neq \omega_\varphi \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$ satisfying
de Rham cohomology. The F'-vector space

$$M'_{\mathrm{dR}} := \left(\bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^*(\omega_\varphi \otimes 1)\right) \cap H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$$

has dimension 2^r.

Thus

$$F^r M'_{\mathrm{dR}} := M'_{\mathrm{dR}} \cap F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)} = F'\omega_\varphi.$$

Betti cohomology. Let

$$I' : H^r_{\mathrm{B}}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Q})^{(E)} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)} \otimes_{F'} \mathbf{C}.$$

The $\mathbf{Q}$-vector space

$$M'_{\mathrm{B}} := I'^{-1}(M'_{\mathrm{dR}} \otimes_{F'} \mathbf{C}) \cap H^r_{\mathrm{B}}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Q})^{(E)}$$

has dimension 2^r.

Definition 3.3.2. An element $\omega_\varphi \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$ is said rational if it satisfies the equations above.

Comparison isomorphisms. There exist isomorphisms

$$\mathscr{I} : M'_{\mathrm{B}} \xrightarrow{\sim} \bigotimes_{j=1}^{r} M_{\mathrm{B},\tau_j},$$

$$\mathscr{J} : M'_{\mathrm{dR}} \xrightarrow{\sim} \left(\bigotimes_{j=1}^{r}(M_{\mathrm{dR}} \otimes_{F,\tau_j} \overline{\mathbf{Q}})\right)^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')},$$

and

$$I_{\tau_j} : M_{\mathrm{B},\tau_j} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{F,\tau_j} \mathbf{C}.$$

Set $I = \bigotimes_{j=1}^{r} I_{\tau_j}$. We have

$$(\star) \qquad I \circ (\mathscr{I} \otimes_{\mathbf{Q}} \mathrm{id}_{\mathbf{C}}) = (\mathscr{J} \otimes_{F'} \mathrm{id}_{\mathbf{C}}) \circ I' : M'_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \bigotimes_{j=1}^{r} \left(M_{\mathrm{dR}} \otimes_{F,\tau_j} \mathbf{C}\right).$$

As in Proposition 3.2.1 we have

Proposition 3.3.3. *Let $\omega_\varphi \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_H(G/Z,X)/F')^{(E)}$ be rational. If E is modular and if Yoshida's conjecture is true, then there exist $\alpha \in F'^{\times}$ and $\xi \in \mathbf{Z} \smallsetminus \{0\}$ such that*

$$\forall \gamma \in H_r(\mathrm{Sh}_H(G,X)(\mathbf{C}),\mathbf{Z}),\ \forall \beta : \prod_{j=2}^{r} \{\pm 1\} \to \{\pm 1\}, \qquad \xi \int_\gamma \alpha \omega_\varphi^\beta \in \Lambda_1 \Omega^\beta.$$

Example. Let $H_1, H_2 \subset \widehat{B}^\times$ be compact open subgroups such that there exists $g \in \widehat{B}^\times$ satisfying $g^{-1}H_1 g \subset H_2$. Let $\omega_{\varphi_2} \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_2}(G/Z,X)/F')^{(E)}$ be rational. Let us explain a way to obtain $\omega_{\varphi_1} \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_1}(G/Z,X)/F')^{(E)}$ rational.

Let

$$\mathrm{pr} : \mathrm{Sh}_{g^{-1}H_1 g}(G/Z,X) \longrightarrow \mathrm{Sh}_{H_2}(G/Z,X)$$

be the map given by

$$[x,b]_{g^{-1}H_1 g} \longmapsto [x,b]_{H_2}$$

and

$$[\cdot g] : \mathrm{Sh}_{H_1}(G/Z,X) \longrightarrow \mathrm{Sh}_{g^{-1}H_1 g}(G/Z,X)$$

by

$$[x,b]_{H_1} \longmapsto [x,bg]_{g^{-1}H_1 g}.$$

Let $\mathrm{pr}_g : \mathrm{Sh}_{H_1}(G/Z,X) \to \mathrm{Sh}_{H_2}(G/Z,X)$ be the composition of pr with $[\cdot g]$.

Choose $\theta_g \in \mathbf{Q}$. Set

$$\omega_{\varphi_1} := \sum_{\substack{g \in \widehat{B}^\times \\ \text{s.t. } g^{-1}H_1 g \subset H_2}} \theta_g \ \mathrm{pr}_g^*(\omega_{\varphi_2}),$$

$$(M'_1)_{\mathrm{dR}} = \left(\sum_g \theta_g \ \mathrm{pr}_g^* \right) (M'_2)_{\mathrm{dR}}$$

and

$$(M'_1)_{\mathrm{B}} = \left(\sum_g \theta_g \ \mathrm{pr}_g^* \right) (M'_2)_{\mathrm{B}}.$$

Proposition 3.3.4. *If $\omega_{\varphi_1} \neq 0$, then the map $\displaystyle\sum_{\substack{g \in \widehat{B}^\times \\ \text{s.t. } g^{-1}H_1 g \subset H_2}} \theta_g \mathrm{pr}_g^*$ is injective on $\bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^*(\omega_{\varphi_2} \otimes 1)$ and $\omega_{\varphi_1} \in F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_1}(G/Z,X)/F')^{(E)}$ is rational.*

Proof. Assume that $\omega = \sum_{\sigma \in \{\pm 1\}^r} \lambda_\sigma t_\sigma^* \omega_{\varphi_2} \in \bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^*(\omega_{\varphi_2} \otimes 1)$ (where $\lambda_\sigma \in \mathbf{C}$) is such that $\sum_g \theta_g \mathrm{pr}_g^*(\omega) = 0$. We have the following equalities :

$$\begin{aligned} \sum_g \theta_g \mathrm{pr}_g^* \omega &= \sum_g \theta_g \mathrm{pr}_g^* \sum_\sigma \lambda_\sigma t_\sigma^* \omega_{\varphi_2} \\ &= \sum_\sigma \lambda_\sigma t_\sigma^* \sum_g \theta_g \mathrm{pr}_g^* \omega_{\varphi_2} \\ \sum_g \theta_g \mathrm{pr}_g^* \omega &= \sum_\sigma \lambda_\sigma t_\sigma^* \omega_{\varphi_1}. \end{aligned}$$

Thus

$$\sum_\sigma \lambda_\sigma t_\sigma^* \omega_{\varphi_1} = 0 \in \bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^* \omega_{\varphi_1},$$

and

$$\forall \sigma \in \{\pm 1\}^r \qquad \lambda_\sigma t_\sigma^* \omega_{\varphi_1} = 0.$$

Hence $\forall \sigma \in \{\pm 1\}^r$ $\lambda_\sigma \in 0$. The map $\sum_{g \in \widehat{B}^\times \text{ s.t. } g^{-1}H_1 g \subset H_2} \theta_g \mathrm{pr}_g^*$ commutes with T_v, $v \notin S$ and is an isomorphism $\bigoplus \mathbf{C} t_\sigma^* \omega_{\varphi_2} \to \bigoplus \mathbf{C} t_\sigma^* \omega_{\varphi_1}$. Hence $\omega_{\varphi_1} \in \left(\bigoplus_{\sigma \in \{\pm 1\}^r} \mathbf{C} t_\sigma^*(\omega_{\varphi_1} \otimes 1) \right) \cap F^r H^r_{\mathrm{dR}}(\mathrm{Sh}_{H_1}(G/Z,X)/F')^{(E)}$ is rational.

$\square$

4. Toric orbits

Let K/F be a quadratic extension satisfying the following properties :

(1) The places $\tau_2, \dots, \tau_r$ of F are split in K.
(2) The places $\tau_1, \tau_{r+1}, \dots, \tau_d$ are ramified in K.

Thanks to the Skolem-Noether theorem, there exists an F-embedding $q : K \hookrightarrow B$, unique up to conjugacy. We will denote by q_j (resp. $\widehat{q}, q_{\mathbf{A}}$) the induced embedding $K \hookrightarrow B_{\tau_j}$ (resp. $\widehat{K} \hookrightarrow \widehat{B}$, $K_{\mathbf{A}} \hookrightarrow B_{\mathbf{A}}$). For each place v of F, set $K_v = K \otimes_F F_v$.

4.1. Cycles on X.

Let $T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)/\mathrm{Res}_{F/\mathbf{Q}}(\mathbf{G}_m)$. Thanks to Hilbert's Theorem 90 we have

$$T(A) = (K \otimes_{\mathbf{Q}} A)^\times/(F \otimes_{\mathbf{Q}} A)^\times$$

for every $\mathbf{Q}$-algebra A.

Fix an embedding $q : T \hookrightarrow G/Z(G)$. The group $T(\mathbf{R})$ is identified with $\prod_{j=1}^d K_{\tau_j}^\times/F_{\tau_j}^\times$ which allows us to define $q_j : K_{\tau_j}^\times/F_{\tau_j}^\times \longrightarrow G_{j,\mathbf{R}}$.

Let $\pi_0(T(\mathbf{R}))$ be the set of connected components of $T(\mathbf{R})$ and denote by $T(\mathbf{R})^\circ$ the component of the identity. Fix a multi-orientation on $T(\mathbf{R})^\circ = \prod_{j=1}^d (K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ$ (i.e. an orientation of each factor $(K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ$) and remark that

$$\pi_0(T(\mathbf{R})) = T(\mathbf{R})/T(\mathbf{R})^\circ \simeq \prod_{j=2}^r \{\pm 1\}.$$

We will focus on the orbits in X under the action of $q(T(\mathbf{R})^\circ)$ by conjugation.

Proposition 4.1.1. *Let $\mathscr{T}^\circ$ be an orbit of $q(T(\mathbf{R})^\circ)$ in X. Then $\mathscr{T}^\circ$ decomposes into a product of orbits in X_j under $q_j(T(\mathbf{R})^\circ)$ and is multi-oriented.*

Proof. The first part of this assertion follows from the natural decomposition $X = X_1 \times \dots X_r$. The orbit $\mathscr{T}^\circ$ decomposes into orbits under $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ)$. For $j = 1$, $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ) \simeq \mathbf{S}^1$ or a point and the orientation does not change. For $j \in \{2, \dots, r\}$, $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ) \simeq \mathbf{R}_+^\times$. The action of $\mathbf{R}_+^\times$ on itself by multiplication does not change the orientation. Hence the multi-orientation induced on $\mathscr{T}^\circ$ by $T(\mathbf{R})^\circ$ is well-defined.

□

In the following sections we shall fix some $q(T(\mathbf{R})^\circ)$-orbit $\mathscr{T}^\circ$ whose projection on X_1 is a point.

Proposition 4.1.2. *$\mathscr{T}^\circ$ is a connected multi-oriented submanifold of real dimension $r - 1$.*

Proof. Recall that $\mathscr{T}^\circ$ is decomposed as $\mathscr{T}^\circ = \{z_1\} \times \mathscr{T}_2 \times \dots \times \mathscr{T}_r$. Fix $x \in X$ such that $\mathscr{T}^\circ = q(T(\mathbf{R})^\circ \cdot x$. Then for $j \in \{2, \dots, r\}$ we have $\mathscr{T}_j = q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ) \cdot \mathrm{pr}_j(x)$. The group $q_j((K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ)$ is naturally identified with $\mathbf{R}_+^\times$ and $\mathscr{T}_j$ is a connected oriented manifold of real dimension one.

□

As a corollary, we have the following decomposition :

$$\mathscr{T}^\circ = \{z_1\} \times \gamma_2 \times \dots \times \gamma_r,$$

when z_1 is one of the two fixed points in the action of $q_1(T(\mathbf{R})^\circ)$ on X_1 and γ_j is an oriented connected submanifold of real dimension one in X_j.

When we use the identification of X with $(\mathbf{C} \smallsetminus \mathbf{R})^r$, the action of $T(\mathbf{R})$ on X by conjugation is an action of $\mathrm{PGL}_2(\mathbf{R})$ on $(\mathbf{C} \smallsetminus \mathbf{R})^r$ by homography. Let $z \in K \smallsetminus F$. For $j \in \{2, \dots, r\}$ the matrix $q_j(z)$ is hyperbolic with exactly two fixed points in $\mathbf{P}^1(\mathbf{R})$, z_j and z_j'. The manifold γ_j is then a circle arc in the Poincaré upper half-plane joining z_j to z_j' (or a line if $z_j' = \infty$). Figure 1 gives some examples of what could the γ_js be in the case of circle arcs.

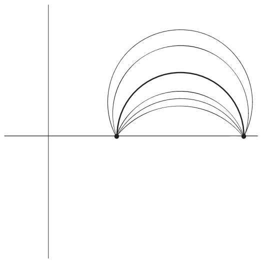


FIGURE 1. Case of circle arcs.

4.2. **Tori on** $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$**.** Let $b \in \widehat{B}^\times$. We will denote by $\mathscr{T}_b^\circ$ the following subset of $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$

$$\mathscr{T}_b^\circ = \left\{ [x,b]_{H\widehat{F}^\times}, x \in \mathscr{T}^\circ \right\}.$$

Proposition 4.2.1. *$\mathscr{T}_b^\circ$ is an oriented torus of real dimension $r-1$.*

Proof. Let $x, x' \in \mathscr{T}^\circ$ and $b \in \widehat{B}^\times$; we know that

$$\begin{aligned}[x,b]_{H\widehat{F}^\times} = [x',b]_{H\widehat{F}^\times} &\iff \exists k \in B^\times \text{ and } h \in H\widehat{F}^\times \qquad (kx', kbh) = (x,b)\\ &\iff \exists k \in B^\times \cap bH\widehat{F}^\times b^{-1} \qquad kx' = x\end{aligned}$$

Since the projection of $\mathscr{T}^\circ$ on X_1 is a point, we have $k \in B \cap q_1(K_{\tau_1}) = q_1(K)$ and

$$k \in q(K^\times) \cap bH\widehat{F}^\times b^{-1}.$$

Thus the stabilizer $\mathscr{W}$ of $\mathscr{T}_b^\circ$ under the action of $q(K^\times)$ is

$$\mathscr{W} = q(K^\times) \cap (bH\widehat{F}^\times b^{-1})$$

which is commensurable with $\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times$. This quotient has rank $r-1$ over $\mathbf{Z}$ as a consequence of Dirichlet's units theorem :

$$\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times \simeq \text{torsion} \times \mathbf{Z}^{r-1},$$

and the torsion is finite. The action of $T(\mathbf{R})^\circ$ on $\mathscr{T}^\circ$ is given by $\prod_{j=2}^r (K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ$ and there is an isomorphism

$$\prod_{j=2}^r (K_{\tau_j}^\times/F_{\tau_j}^\times)^\circ \xrightarrow{\sim} \mathbf{R}^{r-1}.$$

The image $\widetilde{\mathcal{O}}$ of $\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times$ in $\mathbf{R}^{r-1}$ is isomorphic to $\mathbf{Z}^s$ with $s \le r-1$. Denote by $\widetilde{\mathcal{O}}_K^\times$ the image of $\mathcal{O}_K^\times$ in $(K\otimes\mathbf{R})^{\times,\ \mathrm{N}_{K/\mathbf{Q}}=1}$. As

$$\prod_{j\notin\{2,\dots,r\}} K_{\tau_j}^\times/F_{\tau_j}^\times \qquad \text{and} \qquad \frac{(K\otimes\mathbf{R})^{\times,\ \mathrm{N}_{K/\mathbf{Q}}=1}}{\widetilde{\mathcal{O}}_K^\times}$$

are compact, $\mathbf{R}^{r-1}/\widetilde{\mathcal{O}}$ is compact. Thus, the image of $\mathcal{O}_{K,+}^\times/\mathcal{O}_F^\times$ in $\mathbf{R}^{r-1}$ is a lattice.

The set $\mathscr{T}_b^\circ$ is a principal homogeneous space under

$$q(K^\times)/\mathscr{W} \simeq (\mathbf{R}/\mathbf{Z})^{r-1}.$$

It is a real torus in $\mathrm{Sh}_H(G/Z,X)(C)$ of dimension $r-1$, which is oriented by the fixed multi-orientation on $\mathscr{T}^\circ$.

□

For each $u \in \pi_0(T(\mathbf{R}))$ and $b \in \widehat{B}^\times$ let

$$\mathscr{T}_b^u = \left\{ [q(u)\cdot x, b]_{H\widehat{F}^\times},\ x \in \mathscr{T}^\circ \right\}.$$

It is a real oriented torus of dimension $r-1$.

Proposition 4.2.2. *The set*

$$\{\mathscr{T}_b^u \mid \ b\in \widehat{B}^\times,\ u\in\pi_0(T(\mathbf{R}))\}$$

does not depend on the choice of $q: K\hookrightarrow B$.

Proof. Let $\tilde{q}: K\hookrightarrow B$ be another embedding. Thanks to the Skolem-Noether theorem there exists $\alpha\in B^\times$ such that

$$\forall k\in K \qquad \tilde{q}(k)=\alpha q(k)\alpha^{-1}.$$

Let $x_0\in X$, and assume that $\mathscr{T}^\circ = q(T(\mathbf{R})^\circ)\cdot x_0$. We have $\widetilde{\mathscr{T}^\circ} := \tilde{q}(T(\mathbf{R})^\circ)\cdot\alpha(x_0)=\alpha\cdot\mathscr{T}^\circ$ and for each $u\in\pi_0(T(\mathbf{R}))$

$$\alpha\cdot q(u)\cdot\mathscr{T}^\circ = \tilde{q}(uT(\mathbf{R})^\circ)\cdot\alpha\cdot x_0.$$

Let $b\in\widehat{B}^\times$. As $\alpha\in B^\times$ we have

$$\widetilde{\mathscr{T}_b^u} := \left[\tilde{q}(u)\widetilde{\mathscr{T}^\circ}, b\right]_{H\widehat{F}^\times} = [\alpha\cdot q(u)\cdot\mathscr{T}^\circ, b]_{H\widehat{F}^\times} = \left[q(u)\cdot\mathscr{T}^\circ, \alpha^{-1}\cdot b\right]_{H\widehat{F}^\times} = \mathscr{T}^u_{\alpha^{-1}b}.$$

The map $b\mapsto\alpha^{-1}b$ is a bijection. Thus

$$\{\mathscr{T}_b^u,\ b\in\widehat{B}^\times,\ u\in\pi_0(T(\mathbf{R}))\} = \{\widetilde{\mathscr{T}_b^u},\ b\in\widehat{B}^\times,\ u\in\pi_0(T(\mathbf{R}))\}.$$

□

Action of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. Let us denote by K^{ab} the maximal abelian extension of K and by $\mathrm{rec}_K: K_{\mathbf{A}}^\times/K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$ the reciprocity map normalized by letting uniformizers correspond to geometric Frobenius elements.

The group $K_{\mathbf{A}}^\times$ acts on $\{\mathscr{T}_b^u \mid b\in\widehat{B}^\times,\ u\in\pi_0(T(\mathbf{R}))\}$ by

$$\forall a=(a_\infty,a_f)\in K_{\mathbf{A}}^\times = K_\infty^\times\times\widehat{K}^\times\ \forall b\in\widehat{B}^\times \qquad a\cdot\mathscr{T}_b^u = \mathscr{T}^{q(a_\infty)u}_{\widehat{q}(a_f)b}.$$

The action of $k\in K^\times$ is trivial; as $q(k)\in B^\times$, the definition of $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$ gives:

$$k\cdot\mathscr{T}_b^u = [q(k)q(u)\mathscr{T}^\circ, \widehat{q}(k)b]_{H\widehat{F}^\times} = [q(u)\mathscr{T}^\circ, b]_{H\widehat{F}^\times} = \mathscr{T}_b^u.$$

The action of $F_{\mathbf{A}}^\times$ is trivial. For $a=(a_\infty,a_f)\in F_{\mathbf{A}}^\times$, and $b\in\widehat{B}^\times$, $\widehat{q}(a_f)b = b\widehat{q}(a_f)$ and $q(a_\infty)q(u)\mathscr{T}^\circ = q(u)\mathscr{T}^\circ$ hence

$$a\cdot\mathscr{T}_b^u = [q(a_\infty)q(u)\mathscr{T}^\circ, \widehat{q}(a_f)b]_{H\widehat{F}^\times} = [q(u)\mathscr{T}^\circ, b]_{H\widehat{F}^\times} = \mathscr{T}_b^u.$$

4.3. **Special cycles on** $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$.

In this section we construct some r-chain on $\mathrm{Sh}_H(G/Z,X)(\mathbf{C})$.

Proposition 4.3.1. *The homology class* $[\mathscr{T}_b^\circ]\in H_{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})$ *of* $\mathscr{T}_b^\circ$ *is torsion.*

Proof. Let us denote by pr the map

$$\mathrm{pr}: X\times\{b\}\longrightarrow \mathrm{Sh}_H(G/Z,X)(\mathbf{C}).$$

$\mathscr{T}_b^\circ$ is in the image of pr and

$$\mathrm{pr}^{-1}(\mathscr{T}_b^\circ) = (\{z_1\}\times\gamma_2\times\cdots\times\gamma_r)\times\{b\}.$$

Let $\omega\in H^{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C})$. Thanks to the Matsushima-Shimura theorem, $\omega=\omega_{\mathrm{univ}}+\omega_{\mathrm{cusp}}$. As $r-1\neq r$ we know that $\omega=\omega_{\mathrm{univ}}$.

- If $r-1$ is odd, then $H^{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C})=\{0\}$.
- If $r-1=2s$ is even, ω is the pull-back of $\bigwedge_{j=2}^r\omega^{(j)}$, where

$$\omega^{(j)}=1 \qquad\text{or}\qquad \frac{\mathrm{d}x_j\wedge\mathrm{d}y_j}{y_j^2}.$$

With the notations of the proof of Proposition 4.2.1, $\mathscr{T}_b^\circ$ is a principal homogeneous space under $\mathscr{W}$. Fix a fundamental domain $\widetilde{\mathscr{W}}$ of $\mathscr{W}$ in $\gamma_2\times\cdots\times\gamma_r$. The incompatibility of degrees gives

$$\int_{\mathscr{T}_b^\circ}\omega = \int_{\widetilde{\mathscr{W}}}\omega^{(2)}\wedge\cdots\wedge\omega^{(r)} = 0,$$

$$\forall \omega \in H^{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C}) \qquad \int_{\mathscr{T}_b^\circ} \omega = 0.$$

This proves that $[\mathscr{T}_b^\circ] = 0 \in H_r(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{C})$ and $[\mathscr{T}_b^\circ] \in H_r(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})$ is torsion.

□

Definition 4.3.2. Let $n \in \mathbf{Z}_{>0}$ be the exponent of $H_{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})_{\mathrm{tors}}$. Then

$$n[\mathscr{T}_b^\circ] = \partial \Delta_b^\circ$$

for some piece-wise differentiable r-chain Δ_b°.

Proposition 3.2.1 proves that the value of

$$\left(\frac{1}{\Omega^\beta}\xi\alpha\int_{\Delta_b^\circ}\omega_\varphi^\beta\right) \in \mathbf{C}$$

modulo Λ_1 does not depend on the particular choice of Δ_b°. If $T(\mathbf{R})^\circ$ is fixed, then we have the following proposition.

Proposition 4.3.3. *Let $\mathscr{T}^\circ$ and $\mathscr{T}'^\circ$ be two special cycles such that $\mathrm{pr}_1(\mathscr{T}^\circ) = \mathrm{pr}_1(\mathscr{T}'^\circ) = \{z_1\}$. Assume that $\mathrm{pr}_j(\mathscr{T}^\circ)$ and $\mathrm{pr}_j(\mathscr{T}'^\circ)$ lie in the same connected component of X_j for each $j \in \{2,\dots,r\}$. Let n be the exponent of $H_{r-1}(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})_{\mathrm{tors}}$ and let Δ_b° and $\Delta_b'^\circ$ satisfy*

$$n[\mathscr{T}_b^\circ] = \partial\Delta_b^\circ \qquad \text{and} \qquad n[\mathscr{T}_b'^\circ] = \partial\Delta_b'^\circ.$$

Then we have

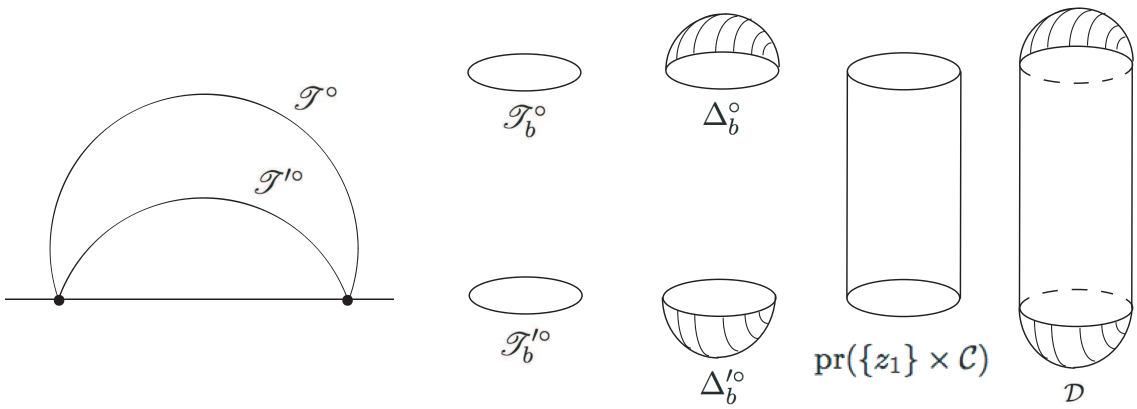
$$\int_{\Delta_b^\circ}\omega_\varphi^\beta = \int_{\Delta_b'^\circ}\omega_\varphi^\beta \pmod{\xi^{-1}\alpha^{-1}\Omega^\beta\Lambda_1}.$$

Proof. Our hypothesis allows us to decompose $\Delta_b'^\circ - \Delta_b^\circ$ into

$$\Delta_b'^\circ - \Delta_b^\circ = \mathrm{pr}(\{z_1\}\times\mathcal{C}) + \mathcal{D},$$

where $\mathcal{D}$ is a cycle with $\partial\mathcal{D} = 0$ and pr is the map

$$\mathrm{pr} : \begin{cases} X & \longrightarrow & \mathrm{Sh}_H(G/Z,X)(\mathbf{C}) \\ x & \longmapsto & [x,b]_{H\widehat{F}^\times} \end{cases}$$



Let us show that $\int_{\Delta_b'^\circ - \Delta_b^\circ}\omega_\varphi^\beta \in \xi^{-1}\alpha^{-1}\Omega^\beta\Lambda_1$.
We have

$$\omega_\varphi^\beta = \sum_\varepsilon \omega_\varepsilon \in \bigoplus_{\varepsilon:\{\tau_1,\dots,\tau_r\}\to\{\pm1\}^r} \Gamma(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),(\Omega_H^{\mathrm{an}})^\varepsilon),$$

Each $\omega_\varepsilon \in \Gamma(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), (\Omega_H^{\mathrm{an}})^\varepsilon)$ satisfies

$$\mathrm{pr}^*(\omega_\varepsilon) = \mathrm{d}z_1 \wedge \omega'_\varepsilon$$

We have

$$\int_{\mathrm{pr}(\{z_1\}\times\mathcal{C})} \omega_\varepsilon = \int_{\{z_1\}\times\mathcal{C}} \mathrm{d}z_1 \wedge \omega'_\varepsilon = 0,$$

thus

$$\int_{\{z_1\}\times\mathcal{C}} \omega_\varphi^\beta = 0.$$

Thanks to Proposition 3.2.1 we have

$$\int_{\mathcal{D}} \omega_\varphi^\beta \in \xi^{-1}\alpha^{-1}\Omega^\beta\Lambda_1$$

and the result follows. □

Corollary 4.3.4. *The value modulo Λ_1 of*

$$\left(\frac{1}{\Omega^\beta}\xi\alpha\int_{\Delta_b^\circ}\omega_\varphi^\beta\right) \in \mathbf{C}$$

depends neither on the choice of $\mathscr{T}^\circ$ whose projection on X_1 is $\{z_1\}$ nor on Δ_b° satisfying $n[\mathscr{T}_b^\circ] = \partial\Delta_b^\circ$.

Definition 4.3.5. We set $J_b^\beta = \frac{1}{\Omega^\beta}\xi\alpha\int_{\Delta_b^\circ}\omega_\varphi^\beta \pmod{\Lambda_1} \in \mathbf{C}/\Lambda_1$, the image of $\mathscr{T}_b^\circ$ by an exotic Abel-Jacobi map.

Properties of J_b^β. For each $u \in \pi_0(T(\mathbf{R}))$ let Δ_b^u be some piece-wise differentiable chain satisfying

$$n\left[[q(u)\cdot\mathscr{T}^\circ, b]_{H\widehat{F}^\times}\right] = \partial\Delta_b^u.$$

Proposition 4.3.6. *We have*

$$J_b^\beta = \frac{1}{\Omega^\beta}\xi\alpha\sum_{u\in\pi_0(T(\mathbf{R}))}\beta(u)\int_{\Delta_b^u}\omega_\varphi \pmod{\Lambda_1}.$$

Proof. Let us identify $\pi_0(T(\mathbf{R}))$ with $\prod_{j=2}^r\{\pm1\}$ and assume that the image of $T(\mathbf{R})^\circ$ is $(1,\dots,1)$. Then

$$\omega_\varphi^\beta = \sum_{u\in\pi_0(T(\mathbf{R}))}\beta(u)t_u^*(\omega_\varphi).$$

The chains $t_u\Delta_b^\circ$ and Δ_b^u are in the same connected component. Thus using 4.3.3, we have

$$\int_{t_u\Delta_b^\circ}\omega_\varphi = \int_{\Delta_b^u}\omega_\varphi$$

and the result follows.

□

Recall that $z_1 \in X_1$ is fixed by $q(K_{\tau_1}^\times)$.

Proposition 4.3.7. *Let $\mathscr{T}^\circ$ and $\mathscr{T}'^\circ$ be two $q(T(\mathbf{R})^\circ)$-orbits such that $\mathrm{pr}_1(\mathscr{T}^\circ) = \mathrm{pr}_1(\mathscr{T}'^\circ) = \{z_1\}$. There exists a unique $u \in \pi_0(T(\mathbf{R}))$ such that, for all $j \in \{2,\dots,r\}$,*

$$\mathrm{pr}_j(\mathscr{T}'^\circ) \text{ and } \mathrm{pr}_j(q(u)\cdot\mathscr{T}^\circ)$$

are in the same connected component of X_j.

If $J_b'^\beta \in \mathbf{C}/\Lambda_1$ denotes the value obtained from $\mathscr{T}'^\circ$, we have

$$J_b'^\beta = \beta(u)J_b^\beta.$$

Proof. Let $x, x' \in X$ be such that $\mathscr{T}^\circ = q(T(\mathbf{R})^\circ) \cdot x$ (resp. $\mathscr{T}'^\circ = q(T(\mathbf{R})^\circ) \cdot x'$). There exists $u \in \pi_0(T(\mathbf{R}))$ such that for all $j \in \{1, \ldots, r\}$, $\mathrm{pr}_j(q(u) \cdot x)$ and $\mathrm{pr}_j(x')$ are in the same connected component of X_j. As $\mathscr{T}'^\circ = q(u) \cdot \mathscr{T}^\circ$, the chain $\Delta_b'^\circ$ whose boundary up to torsion is $[\mathscr{T}'^\circ, b]_{H\hat{F}^\times}$, equals Δ_b^u. Thus

$$\sum_{u' \in \pi_0(T(\mathbf{R}))} \beta(u') \int_{\Delta_b'^{u'}} \omega_\varphi = \sum_{u' \in \pi_0(T(\mathbf{R}))} \beta(u') \int_{\Delta_b^{uu'}} \omega_\varphi = \beta(u) \sum_{u'' \in \pi_0(T(\mathbf{R}))} \beta(u'') \int_{\Delta_b^{u''}} \omega_\varphi.$$

□

Let $q, q' : K \hookrightarrow B$ be two embeddings and $x \in X$, $\mathscr{T}^\circ = q(T(\mathbf{R})^\circ) \cdot x$ (resp. $\mathscr{T}'^\circ = q'(T(\mathbf{R})^\circ) \cdot x'$). There exists $a \in B^\times$ such that

$$q' = aqa^{-1}$$

thanks to the Skolem-Noether theorem. For each $j \in \{1, \ldots, r\}$, $\mathrm{pr}_j(\mathscr{T}^\circ)$ and $\mathrm{pr}_j(\mathscr{T}'^\circ)$ are in the same connected component of X_j if and only if $\tau_j(\mathrm{nr}(a)) > 0$.

Using 4.3.7 we obtain

Proposition 4.3.8. *If*

$$\alpha = (\mathrm{sgn} \circ \tau_j(\mathrm{nr}(a)))_{j \in \{1, \ldots, r\}} \in \{\pm 1\}^{r-1},$$

then

$$J_b'^\beta = \beta(\alpha) J_b^\beta.$$

Let $\mathrm{N}_{B^\times}(K^\times)$ be the normalizer of $K^\times$ in $B^\times$. Let $a \in \mathrm{N}_{B^\times}(K^\times) \smallsetminus K^\times$. After multiplying a by an element in $K^\times$ we may assume

$$\forall j \in \{2, \ldots, r\} \qquad \tau_j(\mathrm{nr}(a)) > 0.$$

We have

$$\mathrm{pr}_1(q(a) \cdot \mathscr{T}^\circ) = t_1(z_1)$$

and

$$\forall j \in \{2, \ldots, r\} \qquad \mathrm{pr}_j(q(a) \cdot \mathscr{T}^\circ) = \mathrm{pr}_j(\mathscr{T}^\circ)$$

but the orientations of $\mathrm{pr}_j(q(a) \cdot \mathscr{T}^\circ)$ and $\mathrm{pr}_j(\mathscr{T}^\circ)$ are not the same.

Thus

$$[t_1 \mathscr{T}^\circ, b]_{H\widehat{F}^\times} = [q(a)\mathscr{T}^\circ, b]_{H\widehat{F}^\times} = [\mathscr{T}^\circ, \widehat{q}(a)^{-1} b]_{H\widehat{F}^\times},$$

but the orientations differ by $(-1)^{r-1}$. Hence

Proposition 4.3.9. *The tori $\mathscr{T}_b^\circ$ and $t_1 \mathscr{T}^\circ_{\widehat{q(a)}b}$ are the same up to orientation.*

5. Generalized Darmon's points

5.1. **The main conjecture.** Let $\Phi_1 : \mathbf{C}/\Lambda_1 \xrightarrow{\sim} E_1(\mathbf{C})$ be the Weierstrass uniformization; i.e. the inverse of Φ_1 is the Abel-Jacobi map for the differential η_1. For each $a_\infty \in K_\infty^\times$, fix some r-chain $q(a_\infty) \cdot \Delta_b^\beta$ satisfying $n[q(a_\infty) \cdot \mathscr{T}_b^\beta] = q(a_\infty) \cdot \Delta_b^\beta$ and denote by $\beta(a_\infty)$ the following sign

$$\beta(a_\infty) = \prod_{j=2}^{r} \beta\left(\mathrm{sgn}\left(\prod_{w | \tau_j} a_{\infty,w}\right)\right).$$

Conjecture 5.1.1. *The point*

$$P_b^\beta = \Phi_1\left(\frac{1}{\Omega^\beta} \xi \alpha \int_{\Delta_b^\beta} \omega_\varphi\right) = \Phi_1(J_b^\beta) \in E_1(\mathbf{C})$$

lies in $E(K^{\mathrm{ab}})$ and

$$\forall a = (a_\infty, a_f) \in K_{\mathbf{A}}^\times \qquad \mathrm{rec}_K(a) P_b^\beta = \Phi_1\left(\frac{\xi\alpha}{\Omega^\beta} \int_{q(a_\infty) \cdot \Delta^\beta_{\widehat{q(a_f)}b}} \omega_\varphi\right) = \beta(a_\infty) P^\beta_{\widehat{q(a_f)}b}.$$

Remark 5.1.2. The choice of $z_1 \in X_1^{q_1(K_{\tau_1}^\times)}$ fixes a morphism $h_1 : \mathbf{S} \longrightarrow G_{1,\mathbf{R}}$, hence a morphism $\mathbf{C}^\times = \mathbf{S}(\mathbf{R}) \longrightarrow G_{1,\mathbf{R}}(\mathbf{R}) = B_{\tau_1}^\times = (B \otimes_{F,\tau_1} \mathbf{R})^\times$ satisfying $h_1(\mathbf{C}^\times) = q_1(K_{\tau_1}^\times)$. This fixes an embedding $\tau_{1,K} : K \hookrightarrow \mathbf{C}$ such that the following diagram

$$\begin{tikzcd}
\mathbf{C}^\times \arrow[r, "h_1"] & (B \otimes_{F,\tau_1} \mathbf{R})^\times \\
 & (K \otimes_{F,\tau_1} \mathbf{R})^\times \arrow[ul, "\tau_{1,K}"'] \arrow[u, "q_1"']
\end{tikzcd}$$

commutes. We may fix $\tilde{\tau}_1 : K^{\mathrm{ab}} \hookrightarrow \mathbf{C}$ above $\tau_{1,K}$, such that

$$\begin{tikzcd}
F \arrow[r, "\tau_1"] \arrow[d] & \mathbf{R} \arrow[r] & \mathbf{C} \\
K \arrow[rr] \arrow[urr, "\tau_{1,K}"] & & K^{\mathrm{ab}} \arrow[u, dotted, "\tilde{\tau}_1"']
\end{tikzcd}$$

commutes. Moreover the isomorphism

$$\begin{cases} \mathrm{Gal}(K^{\mathrm{ab}}/K) & \xrightarrow{\sim} & \mathrm{Gal}(\tilde{\tau}_1(K^{\mathrm{ab}})/\tau_{1,K}(K)) \\ \sigma & \longmapsto & \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1} \end{cases}$$

does not depend on the choice of $\tilde{\tau}_1$. If $\tilde{\tau}_1'$ is another embedding above $\tau_{1,K}$, then $\tilde{\tau}_1' = \tilde{\tau}_1 \circ \sigma'$ with $\sigma' \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and

$$\forall \sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K) \qquad \tilde{\tau}_1' \circ \sigma \circ \tilde{\tau}_1'^{\,-1} = \tilde{\tau}_1 \circ \sigma' \sigma \sigma'^{\,-1} \circ \tilde{\tau}_1^{-1} = \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1}$$

because $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is commutative. Hence the Galois action of 5.1.1 does not depend on the particular choice of $\tilde{\tau}_1$.

Remark 5.1.3. Using conjecture 5.1.1, we obtain

$$\forall a_\infty \in K_\infty^\times \qquad \mathrm{rec}_K(a_\infty) P_b^\beta = \beta(a_\infty) P_b^\beta.$$
$$\forall a \in F_{\mathbf{A}}^\times \qquad \mathrm{rec}_K(a) P_b^\beta = P_b^\beta.$$

5.2. **Field of definition.** Let $B_+^\times = \{b \in B^\times \mid \forall j \in \{2, \dots, r\},\ \tau_j(\mathrm{nr}(b)) > 0\}$. It is diagonally embedded in $(B \otimes \mathbf{R})^\times$. Set

$$K_b^+ = (K^{\mathrm{ab}})^{\mathrm{rec}_K(q_{\mathbf{A}}^{-1}(bH\hat{F}^\times b^{-1} B_+^\times))} \quad \text{and} \quad K_b := (K^{\mathrm{ab}})^{\mathrm{rec}_K(q_{\mathbf{A}}^{-1}(bH\hat{F}^\times b^{-1} B^\times))} \subset K_b^+.$$

Note that K_b and K_b^+ depend on the choice of $q : K \hookrightarrow B$.

Proposition 5.2.1. *The point P_b^β is defined over K_b^+ : $P_b^\beta \in E(K_b^+)$.*

Proof. Let $a = (1_\infty, bhfb^{-1})(a_\infty, 1_f) \in q_{\mathbf{A}}^{-1}(bH\hat{F}^\times b^{-1} B_+^\times)$ with $f \in \hat{F}^\times$ and $h \in H$. We have

$$\mathrm{rec}(a) P_b^\beta = \mathrm{rec}(q_{\mathbf{A}}^{-1}((1_\infty, bhfb^{-1}))) P_b^\beta = P_{bhfb^{-1}b}^\beta = P_{bhf}^\beta = P_b^\beta$$

□

Remark that rec_K induces a surjection

$$\mathcal{R} : \pi_0(T(\mathbf{R})) = \frac{(K \otimes_{\mathbf{Q}} \mathbf{R})^\times}{(F \otimes_{\mathbf{Q}} \mathbf{R})^\times (K \otimes_{\mathbf{Q}} \mathbf{R})_+^\times} \simeq \prod_{j=2}^r \{\pm 1\} \twoheadrightarrow \mathrm{Gal}(K_b^+/K_b).$$

Thus, we have

Proposition 5.2.2. *The points P_b^β lie in $K_b^\beta = (K_b^+)^{\mathcal{R}(\mathrm{Ker}\ \beta)}$.*

Remark 5.2.3. As $\mathrm{Ker}\beta$ has index 2 in $\prod_{j=2}^r \{\pm 1\}$, the field K_b^β has degree 1 or 2 over K_b.

Assume that the conductor N of E decomposes as $N = N_+N_-$ with $N = \mathfrak{p}_1 \dots \mathfrak{p}_t$, $\mathfrak{p}_i$ distinct prime ideals of $\mathcal{O}_F$ and $t \equiv d - r \mod 2$. If $\mathrm{Ram}(B) = \{\tau_{r+1}, \dots, \tau_d\} \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ and $H = (R \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times$ where $R \subset B$ is an Eichler order of level N_+, then K_b is a ring class field of conductor $\mathfrak{f}_b$ and K_b^+ a ring class field of conductor $\mathfrak{f}_b\mathfrak{f}_\infty$, where $\mathfrak{f}_\infty = \prod_{j=2}^r \tau_j$.

5.3. **Local invariants of B.** Let π be the irreductible automorphic representation of $B_{\mathbf{A}}^\times$ generated by φ and

$$\eta_K = \eta_{K/F} : F_{\mathbf{A}}^\times / F^\times \mathrm{N}_{K/F}(K_{\mathbf{A}}^\times) \longrightarrow \{\pm 1\}$$

the quadratic character of K/F. For each place v of F let $\mathrm{inv}_v(B_v) \in \{\pm 1\}$ be the invariant of B: $\mathrm{inv}_v(B_v) = 1$ if and only if $B_v \simeq M_2(F_v)$.

Fix $b \in \widehat{B}^\times$ and a character

$$\chi : \mathrm{Gal}(K_b^+/K) \longrightarrow \mathbf{C}^\times,$$

which will be identified with

$$K_{\mathbf{A}}^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \longrightarrow \mathrm{Gal}(K_b^+/K) \xrightarrow{\chi} \mathbf{C}^\times.$$

Let $L(\pi \times \chi, s)$ be the Rankin-Selberg L function, see [Jac72] page 132 and [JL70] section 12. This function admits, since π has trivial central character, a holomorphic extension to $\mathbf{C}$ satisfying

$$L(\pi \times \chi, s) = \varepsilon(\pi \times \chi, s) L(\pi \times \chi, 1 - s).$$

In this section, we prove the following

Proposition 5.3.1. *Let $b \in \widehat{B}^\times$ and assume conjecture 5.1.1. If*

$$e_{\overline{\chi}}(P_b^\beta) = \sum_{\sigma \in \mathrm{Gal}(K_b^+/K)} \chi(\sigma) \otimes P_b^\beta \in E(K_b^+) \otimes \mathbf{Z}[\chi]$$

is not torsion, then $\beta = \chi_\infty$,

$$\forall v \neq \tau_1 \qquad \eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \mathrm{inv}_v(B_v) \quad \text{and} \quad \varepsilon(\pi \times \chi, \frac{1}{2}) = -1.$$

We shall use the following theorem ([Tun83] and [Sai93]).

Theorem 5.3.2. *The equality $\eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \mathrm{inv}_v(B_v)$ holds if and only if there exists a non-zero invariant linear form*

$$\ell_v : \pi_v \times \chi_v \longrightarrow \mathbf{C}$$

unique up to a scalar satisfying

$$\forall a \in K_v^\times \ \forall u \in \pi_v \qquad \ell_v(q_v(a)u) = \chi_v(a)^{-1}\ell_v(u)$$

i.e. ℓ_v is $q(K_v^\times)$-invariant.

Proof. (of Proposition 5.3.1) We follow the proof of [AN10], Proposition 2.6.2.

Let S' be a finite set of finite places of F containing the places where B, π or K_b^+/F ramify, and such that the map $r = (r_v : K_v^\times \longrightarrow \mathrm{Gal}(K_b^+/K))_{v \in S'}$ obtained by composition

$$r : \prod_{v \in S'} K_v^\times \longrightarrow K_{\mathbf{A}}^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \longrightarrow \mathrm{Gal}(K_b^+/K)$$

is surjective.

For each $v \in S'$ let

$$j_v : \begin{cases} K_v & \hookrightarrow & B_v \\ k & \longmapsto & b_v^{-1} q_v(k) b_v \end{cases}$$

and

$$j = (j_v)_{v \in S'} : \prod_{v \in S'} K_v \hookrightarrow \prod_{v \in S'} B_v.$$

As S' does not contain any archimedean place of F,

$$\forall a \in \prod_{v \in S'} K_v^\times \qquad [\mathscr{T}^\circ, \widehat{q}(a)b]_{H\hat{F}^\times} = [\mathscr{T}^\circ, bj(a)]_{H\hat{F}^\times}$$

and

$$\forall a \in \prod_{v \in S'} K_v^\times \ \forall b \in \widehat{B}^\times \qquad \mathrm{rec}_K(a) P_b^\beta = P_{\widehat{q}(a)b}^\beta = P_{bj(a)}^\beta.$$

Let $(K_v^\times)^\circ \subset K_v^\times$ be the inverse image of $(K_v^\times/\mathcal{O}_{K,v}^\times)^{\mathrm{Gal}(K/F)} \subset K_v^\times/\mathcal{O}_{K,v}^\times$.

We have

$$K_v^\times/\mathcal{O}_{K,v}^\times F_v^\times \xrightarrow{\sim} \begin{cases} 0 & \text{if } v \text{ is inert in } K/F \\ \mathbf{Z}/2\mathbf{Z} & \text{if } v \text{ ramifies in } K/F \\ \mathbf{Z} & \text{if } v \text{ splits in } K/F, \end{cases}$$

the quotient $(K_v^\times)^\circ/F_v^\times$ is compact and

$$D_v := K_v^\times/(K_v^\times)^\circ \xrightarrow{\sim} \begin{cases} \mathbf{Z} & \text{if } v \text{ splits in } K/F \\ 0 & \text{otherwise,} \end{cases}$$

$$(K_v^\times)^\circ/\mathcal{O}_{K,v}^\times F_v^\times \xrightarrow{\sim} \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } v \text{ ramifies in } K/F \\ 0 & \text{otherwise.} \end{cases}$$

For each $v \in S'$, $C_v = \mathcal{O}_{K,v}^\times \cap \mathrm{Ker}(r_v)$ is an open subgroup of $\mathcal{O}_{K,v}^\times$ and $V_v^\circ = (K_v^\times)^\circ/F_v^\times C_v$ is finite.

Let V_v be the following subset of $K_v^\times/F_v^\times C_v$:

- if v does not split in K/F, $V_v^\circ = K_v^\times/F_v^\times C_v$ and $V_v := V_v^\circ$.
- If v splits in K/F, we fix some section of $K_v^\times \twoheadrightarrow K_v^\times/(K_v^\times)^\circ \xrightarrow{\sim} \mathbf{Z}$. Hence $K_v^\times = (K_v^\times)^\circ \times D_v$ and there exists $n_v \geq 1$ such that $\mathrm{Ker}(r_v\mid_{D_v}) = n_v D_v$.
  Fix a set of representatives $D_v' \subset D_v$ of D_v/n_vD_v and set $V_v = V_v^\circ D_v' \subset K_v^\times/F_v^\times C_v$.

Let $V = \prod_{v\in S'} V_v \subset \prod_{v\in S'} K_v^\times/F_v^\times C_v$, which is stable under multiplication by the abelian group $V^\circ = \prod_{v\in S'} V_v^\circ$ and such that $V \hookrightarrow \prod_{v\in S'} K_v^\times/F_v^\times C_v \xrightarrow{r} \mathrm{Gal}(K_b^+/K)$ is surjective with fibers of cardinality $\frac{|V|}{|\mathrm{Gal}(K_b^+/K)|}$. We have

$$\begin{aligned} \frac{|V|}{|\mathrm{Gal}(K_b^+/K)|} e_{\overline{\chi}}(P_b^\beta) &= \frac{|V|}{|\mathrm{Gal}(K_b^+/K)|} \sum_{\sigma\in\mathrm{Gal}(K_b^+/K)} \chi(\sigma)\otimes\sigma\cdot P_b^\beta \\ &= \sum_{a\in V} \chi(a)\otimes P_{bj(a)}^\beta. \end{aligned}$$

Fix some open-compact subgroup $H_1 \subset \bigcap_{a\in V} j(a)Hj(a)^{-1}$. Using the maps

$$\mathrm{Sh}_{H_1}(G/Z,X) \xrightarrow{[\cdot j(a)]} \mathrm{Sh}_{j(a)^{-1}H_1j(a)}(G/Z,X) \xrightarrow{\mathrm{pr}} \mathrm{Sh}_H(G/Z,X),$$

we have

$$\begin{aligned} \sum_{a\in V}\chi(a)\int_{\Delta_{bj(a)}^\circ}\omega_\varphi^\beta &= \sum_{a\in V}\chi(a)\int_{\Delta_b^\circ}[\cdot j(a)]^*\omega_\varphi^\beta \\ &= \int_{\Delta_b^\circ}\sum_{a\in V}\chi(a)[\cdot j(a)]^*\omega_\varphi^\beta \\ &= \int_{\Delta_b^\circ}\omega_1^\beta, \end{aligned}$$

where

$$\omega_1^\beta := \sum_{a\in V}\chi(a)[\cdot j(a)]^*\omega_\varphi^\beta.$$

Whenever $\frac{|V|}{|\mathrm{Gal}(K_b^+/K)|}e_{\overline{\chi}}(P_b^\beta) = \sum_{a\in V}\chi(a)\otimes P_{bj(a)}^\beta \in \mathbf{Z}[\chi]\otimes_{\mathbf{Z}} E(K_b^+) \subset \mathbf{Z}[\chi]\otimes_{\mathbf{Z}}\mathbf{C}/\Lambda_1$ is not torsion, there exists $\sigma : \mathbf{Z}[\chi]\hookrightarrow\mathbf{C}$ such that

$$\frac{\xi\alpha}{\Omega^\beta}\int_{\Delta_b^\circ}\sum_{a\in V}{}^\sigma\chi(a)[\cdot j(a)]^*\omega_\varphi^\beta \notin \mathbf{Q}[{}^\sigma\chi]\cdot\Lambda_1,$$

where ${}^\sigma\chi = \sigma\circ\chi$. The vector

$${}^\sigma\omega_1 = \sum_{a\in V}{}^\sigma\chi(a)[\cdot j(a)]^*\omega_\varphi \in \pi^{H_1}\cap\Gamma(\mathrm{Sh}_{H_1}(G/Z,X),\Omega_{H_1})$$

is non-zero and invariant under $j(\prod_{v\in S'}(K_v^\times)^\circ)$. Moreover,

$$\forall a\in\prod_{v\in S'}(K_v^\times)^\circ \qquad j(a)\omega_1 = {}^\sigma\chi^{-1}(a)\omega_1.$$

Let

$$^{\sigma}\ell_{S'}:\bigotimes_{v\in S'}{}^{\sigma}\pi_v=\bigotimes_{v\in S'}\pi_v\longrightarrow \mathbf{C}(^{\sigma}\chi^{-1})$$

be the $j(\prod_{v\in S'}(K_v^{\times})^{\circ})$-invariant projection on $\mathbf{C}\omega_1$.

Assume that $v\in S'$ does not split in K. In this case $(K_v^{\times})^{\circ}=K_v^{\times}$ and $^{\sigma}\ell_{S'}$ induces a $q_v(K_v^{\times})$-invariant linear form $^{\sigma}\ell_v:\pi_v\to\mathbf{C}(^{\sigma}\chi_v^{-1})$. We have $^{\sigma}\ell_v(\omega_{1,v})\neq 0$, where

$$\omega_{1,v}=\sum_{a_v\in V_v}{}^{\sigma}\chi\circ r_v(a_v)[\cdot j_v(a_v)]^*\omega_{\varphi}.$$

As $\varepsilon_v(\pi_v\times^{\sigma}\chi_v,\frac12)$ is independent of $\sigma:\mathbf{Z}[\chi]\hookrightarrow\mathbf{C}$, Theorem 5.3.2 shows that

$$\eta_{K,v}(-1)\varepsilon(\pi_v\times\chi_v,\frac{1}{2})=\mathrm{inv}_v(B_v).$$

When $v\in S'$ splits in K or $v\notin S'\cup S_{\infty}$, the equality

$$\eta_{K,v}(-1)\varepsilon(\pi_v\times\chi_v,\frac{1}{2})=1=\mathrm{inv}_v(B_v)$$

follows from calculations which may be found for example in [Nek06] Proposition 12.6.2.4.
Global sign. If $v=\tau_j$ is an archimedean place, then $\varepsilon(\pi_v\times\chi_v,\frac12)=1$. Moreover $\eta_{K,v}(-1)=1$ if and only if $j\in\{2,\dots,r\}$ and $\mathrm{inv}_v(B_v)=1$ if and only if $j\in\{1,\dots,r\}$. Thus

$$\eta_{K,v}(-1)\mathrm{inv}_v(B_v)=\begin{cases}-1\times 1 & \text{if } j=1\\ 1\times 1 & \text{if } j\in\{2,\dots,r\}\\ -1\times -1 & \end{cases}$$

and

$$\forall j\in\{1,\dots,d\}\quad \varepsilon_v(\pi_v\times\chi_v,\frac{1}{2})=\eta_{K,v}(-1)\mathrm{inv}_v(B_v)\times\begin{cases}-1 & \text{if } j=1\\ 1 & \text{if } j>1.\end{cases}$$

Hence

$$\varepsilon(\pi\times\chi,\frac{1}{2})=-\prod_v\eta_{K,v}(-1)\mathrm{inv}_v(B_v)=-1.$$

□

5.4. **Global invariant linear form and a conjectural Gross-Zagier formula.** For any open subgroup $H'\subset H$, $b\in\widehat{B}^{\times}$ and $u\in\pi_0(T(\mathbf{R}))$ fix $\Delta^u_{H',b}\in C^r(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Q})$ such that $\partial\Delta^u_{H',b}=[\mathscr{T}^u_{H',b}]$, where $\mathscr{T}^u_{H',b}=\{[q(u)x,b]_{H'\widehat{F}^{\times}},\ x\in\mathscr{T}^{\circ}\}$.

Recall that

$$\forall u'\in\pi_0(T(\mathbf{R}))\qquad t_{u'}\Delta^u_{H,b}=\Delta^{uu'}_b.$$

Let π_∞ be the archimedean part of π. Fix $\varphi_\infty\in\pi_\infty$ a lowest weight vector of weight $(\underbrace{2,\dots,2}_{r},0,\dots,0)$ of π_∞ and ω_φ such that $\omega_\varphi=\varphi_\infty\otimes\varphi_f\in\pi_\infty\otimes\pi_f\subset S_2(B^{\times}_{\mathbf{A}})$.

Let us denote by ${}_{\mathbf{Q}}\pi_f$ the sub $\mathbf{Q}[\widehat{B}^{\times}]$-module of π_f generated by φ_f.

Proposition 5.4.1. *The space ${}_{\mathbf{Q}}\pi_f$ is a $\mathbf{Q}$-vector space and ${}_{\mathbf{Q}}\pi_f\otimes_{\mathbf{Q}}\mathbf{C}\longrightarrow\pi_f$ is surjective.*

Proof. The space $\mathrm{Im}({}_{\mathbf{Q}}\pi_f\otimes_{\mathbf{Q}}\mathbf{C}\to\pi_f)$ is a zero subvector space of π_f invariant under $B^{\times}_{\mathbf{A}}$. As π_f is irreducible, we have $\mathrm{Im}({}_{\mathbf{Q}}\pi_f\otimes_{\mathbf{Q}}\mathbf{C}\to\pi_f)=\pi_f$ and ${}_{\mathbf{Q}}\pi_f\otimes_{\mathbf{Q}}\mathbf{C}\to\pi_f$ is surjective. □

Fix $\eta\neq 0\in H^0(E,\Omega_{E/F})$. There exists $\alpha\in F'^{\times}$ such that

$$\mathscr{J}(\alpha\omega_\varphi)=\eta.$$

Fix a continuous character of finite order $\chi:K^{\times}_{\mathbf{A}}/K^{\times}F^{\times}_{\mathbf{A}}\longrightarrow\mathbf{Z}[\chi]^{\times}$. Let $H'\subset H$ be any open compact subgroup of $\widehat{B}^{\times}$ satisfying $\chi\left(q_{\mathbf{A}}^{-1}(H'F^{\times}_{\mathbf{A}})\right)=1$. Assume that there exists $b_0\in\widehat{B}^{\times}$ such that $b_0^{-1}H'b_0\subset H$. Let pr_{b_0} be the map $\mathrm{Sh}_{H'}(G/Z,X)\to\mathrm{Sh}_H(G/Z,X)$ defined on complex points by

$$[x,b]_{H'\widehat{F}^{\times}}\mapsto[x,bb_0]_{H\widehat{F}^{\times}}.$$

Proposition 5.4.2. *If $b_0^{-1}H'b_0 \subset H$ for some $b_0 \in \widehat{B}^\times$, then*

$$\forall Z' \in C^r(\mathrm{Sh}_{H'}(G/Z,X)(\mathbf{C}),\mathbf{Z}) \qquad \int_{Z'} \mathrm{pr}_{b_0}^*(\omega_\varphi^{\chi_\infty}) \in \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1.$$

Proof. Let $Z = \mathrm{pr}_{b_0}(Z') \in C^r(\mathrm{Sh}_H(G/Z,X)(\mathbf{C}),\mathbf{Z})$. We have

$$\int_{Z'} \mathrm{pr}_{b_0}^*\omega_\varphi^{\chi_\infty} = \deg(\mathrm{pr}_{b_0} : Z' \to Z)\int_Z \omega_\varphi^{\chi_\infty}.$$

Thanks to Proposition 3.3.3, we have $\int_Z \omega_\varphi^{\chi_\infty} \in \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1$ hence $\int_{Z'} \mathrm{pr}_{b_0}^*\omega_\varphi^{\chi_\infty} \in \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1$. □

Denote by $\mathrm{pr} : \mathrm{Sh}_{H'}(G/Z,X) \longrightarrow \mathrm{Sh}_H(G/Z,X)$ the natural projection, and by $(K\otimes\mathbf{R})_+^\times$ the set of elements in $(K\otimes\mathbf{R})^\times$ whose norm to F is positive at each place of F. We have $\pi_0(T(\mathbf{R})) = \frac{(K\otimes\mathbf{R})^\times}{(F\otimes\mathbf{R})^\times(K\otimes\mathbf{R})_+^\times}$.
The following formula

$$\ell_\chi(\omega') = \frac{1}{[H:H']\deg(\mathscr{T}_{H',b} \xrightarrow{\mathrm{pr}} \mathscr{T}_{H,b})} \sum_{a \in \frac{K_{\mathbf{A}}^\times}{q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' \pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1},$$

where $\partial\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)} = [\mathscr{T}^{q(a_\infty)}_{H',\widehat{q}(a_f)}]$, is independent of the specific choice of $\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}$: we can assume that $\omega' = \mathrm{pr}_{b_0}^*(\omega_\varphi)$ for some $b_0 \in \widehat{B}^\times$; decompose each $a \in K_{\mathbf{A}}^\times/q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times$ as $a = (a_f, 1_\infty)(1_f, a_\infty)$. Remark that

$$K_{\mathbf{A}}^\times/q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times = \widehat{K}^\times/\widehat{q}^{-1}(H'\widehat{F}^\times) \times (K\otimes\mathbf{R})^\times/(K\otimes\mathbf{R})_+^\times,$$

hence $a_f \in \widehat{K}^\times/\widehat{q}^{-1}(H'\widehat{F}^\times)$ and $a_\infty \in (K\otimes\mathbf{R})^\times/(K\otimes\mathbf{R})_+^\times$.
Thanks to Proposition 5.4.2, the following formula

$$\begin{aligned}\sum_{a_\infty \in K_\infty^\times} \chi_\infty(a_\infty)\int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega' &= \sum_{a_\infty \in K_\infty^\times} \chi_\infty(a_\infty)\int_{\Delta_{H',\widehat{q}(a_f)}} t_{q(a_\infty)}\mathrm{pr}_{b_0}^*\omega_\varphi \\ &= \int_{\Delta_{H,\widehat{q}(a_f)}} \omega_\varphi^{\chi_\infty} \qquad \pmod{\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\end{aligned}$$

does not depend on the specific choice of $\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}$.
Thus, the expression of $\ell_\chi(\omega')$ above defines a linear form

$$\ell_\chi : S_2^{H'} \cap \mathbf{Q}[\widehat{B}^\times]\omega_\varphi \longrightarrow \mathbf{Q}(\chi)\otimes_{\mathbf{Q}} (\mathbf{C}/\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1).$$

To simplify the notations, let

$$\delta_{H',H} = \deg(\mathscr{T}_{H',b} \xrightarrow{\mathrm{pr}} \mathscr{T}_{H,b}) \qquad \text{and} \qquad W_{H'} = K_{\mathbf{A}}^\times/q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times.$$

Thus

$$\ell_\chi(\omega') = \frac{1}{[H:H']\delta_{H',H}} \sum_{a \in W_{H'}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q}(a_f)}} \omega'.$$

Proposition 5.4.3. *(1) Let $H'' \subset H' \subset H$ be open compact subgroups such that $\chi(q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)) = 1$ and pr^* the map $\mathrm{pr}^* : S_2^{H'}(B_{\mathbf{A}}^\times) \longrightarrow S_2^{H''}(B_{\mathbf{A}}^\times)$.*
If $\omega' \in S_2^{H'}(B_{\mathbf{A}}^\times) \cap \mathbf{Q}[\widehat{B}^\times]\omega_\varphi$, then $\ell_\chi(\omega') = \ell_\chi(\mathrm{pr}^*(\omega'))$ and ℓ_χ defines a linear form on $\mathbf{Q}[\widehat{B}^\times]\omega_\varphi$.
(2) We have

$$\forall a \in \widehat{K}^\times\ \forall \omega \in \mathbf{Q}[\widehat{B}^\times]\omega_\varphi \qquad \ell_\chi([\cdot\widehat{q}(a_f)]^*\omega) = \chi_f(a)^{-1}\ell_\chi(\omega).$$

(3) If χ factors through $\mathrm{Gal}(K_b^+/K)$ and if $P_b^\beta = \Phi_1\left(\int_{\Delta_{H,b}} \omega_\varphi^\beta\right) \otimes 1 \in \mathbf{C}/\mathbf{Q}\Lambda_1$, then

$$e_{\overline{\chi}}(P_b^{\chi_\infty}) = \sum_{\mathrm{Gal}(K_b^+/K)} \chi(\sigma)\otimes\sigma(P_b^{\chi_\infty}) \in \mathbf{Q}(\chi)\otimes_{\mathbf{Q}} E(K_b^+) \subset \mathbf{Q}(\chi)\otimes_{\mathbf{Q}} (\mathbf{C}/\mathbf{Q}\Lambda_1)$$

equals $\Phi_1(\ell_\chi([\cdot b]^*\omega_\varphi))$, up to a non-zero rational factor.

Proof. **Proof of 1.** Let $a \in \widehat{K}^\times$. We have $\mathrm{pr}(\Delta_{H'',\widehat{q}(a_f)}) = \Delta_{H',\widehat{q}(a_f)}$ and

$$\int_{\Delta_{H'',b}} \mathrm{pr}^*\omega' = \deg(\mathscr{T}_{H'',b} \longrightarrow \mathscr{T}_{H',b}) \int_{\Delta_{H',b}} \omega' = \delta_{H'',H'} \int_{\Delta_{H',b}} \omega'.$$

As $\chi(q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)) = 1$, we have (thanks to Proposition 5.4.2)

$$\begin{aligned}
\ell_\chi(\mathrm{pr}^*\omega') &= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a\in W_{H''}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H'',\widehat{q(a_f)}}} \mathrm{pr}^*\omega' &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \frac{\delta_{H'',H'}}{\delta_{H'',H}} \sum_{a\in W_{H''}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q(a_f)}}} \omega' &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \frac{\delta_{H'',H'}}{[H:H'']\delta_{H'',H}} \sum_{a\in W_{H'}} [H':H'']\chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q(a_f)}}} \omega' &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \frac{[H':H'']}{[H:H'']\delta_{H',H}} \sum_{a\in W_{H'}} \chi(a) \otimes \int_{\Delta^{q(a_\infty)}_{H',\widehat{q(a_f)}}} \omega' &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \ell_\chi(\omega').
\end{aligned}$$

Proof of 2. Assume H'' is sufficiently small such that $[\cdot\widehat{q}(a_f)]^*\mathrm{pr}^*\omega \in S_2^{H''}$. We have

$$\begin{aligned}
\ell_\chi([\cdot\widehat{q}(a_f)]^*\omega) &= \ell_\chi([\cdot\widehat{q}(a_f)]^*\mathrm{pr}^*\omega)\\
&= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a'\in W_{H''}} \chi(a') \otimes \int_{\Delta^{q(a'_\infty)}_{H'',\widehat{q(a')}}} [\cdot\widehat{q}(a_f)]^*\mathrm{pr}^*\omega &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a'\in W_{H''}} \chi(a') \otimes \int_{\Delta^{q(a'_\infty)}_{H'',\widehat{q(aa')}}} \mathrm{pr}^*\omega &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \frac{1}{[H:H'']\delta_{H'',H}} \sum_{a''\in W_{H''}} \chi(a''a^{-1}) \otimes \int_{\Delta^{q(a''_\infty)}_{H'',\widehat{q(a'')}}} \mathrm{pr}^*\omega &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \chi_f(a)^{-1}\frac{1}{[H:H'']\delta_{H'',H}} \sum_{a''\in W_{H''}} \chi(a'') \otimes \int_{\Delta^{q(a''_\infty)}_{H'',\widehat{q(a'')}}} \mathrm{pr}^*\omega &&\pmod{\mathbf{Q}(\chi)\otimes_{\mathbf{Q}} \mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1}\\
&= \chi_f(a)^{-1}\ell_\chi(\mathrm{pr}^*\omega)\\
&= \chi_f(a)^{-1}\ell_\chi(\omega)
\end{aligned}$$

Proof of 3. As $\omega_\varphi \in S_2(B_{\mathbf{A}}^\times) = \bigcup_H S_2^H(B_{\mathbf{A}}^\times)$, there exists H' sufficiently small such that

$$\omega_\varphi \in S_2^{H'} \quad \text{and} \quad [\cdot b]^*\omega_\varphi \in S_2^{H'}.$$

Let $m = [K_{\mathbf{A}}^\times / q_{\mathbf{A}}^{-1}(H'F_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times : \mathrm{Gal}(K_b^+/K)]$ and $\nu = \frac{1}{[H:H']\deg(\mathscr{T}_{H'}\longrightarrow\mathscr{T}_H)}$. We have :

$$\begin{aligned}
\ell_\chi(\circ[\cdot b]^*\omega_\varphi) &= \nu \sum_{a\in \frac{K_{\mathbf{A}}^\times}{q_{\mathbf{A}}^{-1}(HF_{\mathbf{A}}^\times)(K\otimes\mathbf{R})_+^\times}} \chi_f(a_f)\chi_\infty(a_\infty)\otimes\int_{\Delta^{q(a_\infty)}_{H',\widehat{q(a_f)}}}[\cdot b]^*\omega_\varphi \quad (\mathrm{mod}\ \mathbf{Q}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \nu\sum_{a_f}\chi_f(a_f)\otimes\sum_{a_\infty}\chi_\infty(a_\infty)\mathrm{rec}_K(a_f)\cdot\int_{\Delta_{H',b}} t_{\mathrm{rec}_K(a_\infty)}\omega_\varphi \quad (\mathrm{mod}\ \mathbf{Q}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \nu m\sum_{\sigma\in\mathrm{Gal}(K_b^+/K)}\chi(\sigma)\otimes\int_{\Delta_{H',b}}\sum_{a_\infty}\chi_\infty(a_\infty)t_{\mathrm{rec}_K(a_\infty)}\omega_\varphi \quad (\mathrm{mod}\ \mathbf{Q}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1)\\
&= \nu m\sum_{\sigma\in\mathrm{Gal}(K_b^+/K)}\chi(\sigma)\otimes\int_{\Delta_{H',b}}\omega_\varphi^{\chi_\infty} \quad (\mathrm{mod}\ \mathbf{Q}(\chi)\otimes_{\mathbf{Q}}\mathbf{Q}\alpha^{-1}\Omega^{\chi_\infty}\Lambda_1),
\end{aligned}$$

hence

$$e_{\overline{\chi}}(P_b^{\chi_\infty}) = \Phi_1(\ell_\chi([\cdot b]^*\omega_\varphi)).$$

□

Let us consider the Néron-Tate height $h_{\mathrm{NT}} : E(K^{\mathrm{ab}})\times E(K^{\mathrm{ab}}) \longrightarrow \mathbf{R}$ extended to an hermitian form

$$h_{\mathrm{NT}} : E(K^{\mathrm{ab}})\otimes\mathbf{C}\times E(K^{\mathrm{ab}})\otimes\mathbf{C}\longrightarrow\mathbf{C}.$$

Recall the condition

$$\forall v\neq\tau_1\quad \varepsilon(\pi_v\times\chi_v,\frac{1}{2})\eta_{K,v}(-1) = \mathrm{inv}_v(B) \tag{2}$$

from Proposition 5.3.2: if 2 fails, then $P_b^{\chi_\infty}\in E(K^{\mathrm{ab}})$ is torsion.
In general, there should be some $k(b,\omega_\varphi)\in\mathbf{C}$ such that

$$\forall\sigma:\mathbf{Q}(\chi)\hookrightarrow\mathbf{C}\quad h_{\mathrm{NT}}(e_{\sigma\overline{\chi}}(P_b^{\chi_\infty})) = k(b,\omega_\varphi)L'(\pi\times{}^\sigma\chi,\frac{1}{2}),$$

as in Gross-Zagier, Zhang and Yuan-Zhang-Zhang [GZ86, Zha01, YZZ09].
This formula explains the following conjecture :

Conjecture 5.4.4. *Let $K_\chi = (K^{\mathrm{ab}})^{\mathrm{Ker}(\chi)}$ be the extension of K trivializing χ. If*

$$\forall v\neq\tau_1\quad \varepsilon(\pi_v\times\chi_v,\frac{1}{2})\eta_{K,v}(-1) = \mathrm{inv}_v(B),$$

then there exists $b\in\widehat{B}^\times$ such that $k(b,\omega_\varphi)\neq 0$ and we have the following equivalences :

$$\begin{aligned}
\ell_\chi\neq 0 &\Longleftrightarrow \exists b\in B_{\mathbf{A}}^\times \text{ such that } K_\chi\subset K_b^+ \text{ and } e_{\overline{\chi}}(P_b^{\chi_\infty})\in\mathbf{Z}[\chi]\otimes E(K_b^+) \text{ is not torsion}\\
&\Longleftrightarrow \exists\sigma:\mathbf{Q}(\chi)\hookrightarrow\mathbf{C}\qquad L'(\pi\times{}^\sigma\chi,\frac{1}{2})\neq 0\\
&\Longleftrightarrow \forall\sigma:\mathbf{Q}(\chi)\hookrightarrow\mathbf{C}\qquad L'(\pi\times{}^\sigma\chi,\frac{1}{2})\neq 0.
\end{aligned}$$

6. A relation to Kudla's program

The theorem of Gross-Kohnen-Zagier asserts that the positions of the traces to $\mathbf{Q}$ of classical Heegner points are given by the Fourier coefficients of some Jacobi form. The geometric proof of Zagier explained for example in [Zag85] has been recently generalized by Yuan, Zhang and Zhang in [YZZ09] using a result of Kudla-Millson [KM90]. In this section we establish a relation between Darmon's construction and Kudla's program. This is a first step in an attempt to apply the arguments of Zagier [Zag85] and Yuan-Zhang and Zhang's [YZZ09] to Darmon's points.

6.1. **Some computations.** Let us fix a modular elliptic curve E/F of conductor $N = N_+N_-$. Assume $\mathrm{Ram}(B) = \{\tau_{r+1}, \dots, \tau_d\} \cup \{v \mid N_-\}$ and that the quadratic extension K/F satisfies the following hypothesis

$$\forall v \mid N_+ \text{ splits in } K \qquad \forall v \mid N_- \text{ is inert in } K.$$

In particular, the relative discriminant $d_{K/F}$ is prime to N. Let R be an Eichler order of B of level N_+. Identify K with its image in B by q and assume $K \cap R = \mathcal{O}_K$, $H = \hat{R}^\times$ (which implies that $\dim \pi_f^H = 1$).

Recall that h_{z_1} defines an embedding $\tau_{1,K} : K \hookrightarrow \mathbf{C}$ and denote by c the non-trivial element of $\mathrm{Gal}(K/F)$. Assume that Conjecture 5.1.1 is true for $\beta = 1$ and let $P = \mathrm{Tr}_{K_1^+/K} P_1 \in E(K)$.

Proposition 6.1.1. *If ε is the global sign of E/F, i.e. $\Lambda(E/F, s) = \varepsilon\Lambda(E/F, 2-s)$, where Λ is the completed L-function of E/F, then $c(P) = -\varepsilon P$.*

Proof. Assume that $K = F(i)$ and $B = K(j)$, with $i^2 = \mathfrak{a} \in F^\times$, $j^2 = \mathfrak{b} \in F^\times$ and $ij = -ji$. Recall that

$$\mathscr{T}_1^\circ = [\mathscr{T}^\circ, 1]_{H\hat{F}^\times}$$

with $\mathscr{T}^\circ = \{z_1\} \times \gamma_2 \times \dots \times \gamma_r$. Thus

$$c(\mathscr{T}_1^\circ) = [\{t_1 z_1\} \times \gamma_2 \times \dots \times \gamma_r, 1]_{H\hat{F}^\times} = (-1)^{r-1}[j^{-1}(\mathscr{T}^\circ), 1]_{H\hat{F}^\times}$$

and

$$c(\mathscr{T}_1^\circ) = (-1)^{r-1}[\mathscr{T}^\circ, j]_{H\hat{F}^\times}$$

since $j \in B^\times$. This shows that $c(P_1) = (-1)^{r-1}P_j$. We will write P_j using only P_1. We will make the following abuse of language. For each place v of F, j_v shall denote the element $(1, \dots, 1, \underbrace{j_v}_{v}, 1 \dots) \in B_{\mathbf{A}}^\times$ and we will use the following lemma

Lemma 6.1.2. *Let $b \in \widehat{B}^\times$ and v a place of F. When $v \mid N_+$, set $k_v \in K_v^\times$ corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{\mathrm{ord}_v(N_+)} \end{pmatrix}$, where ϖ_v is an uniformizer of K_v. If $b_v = 1$, then*

$$P_{bj_v} = \begin{cases} -\varepsilon_v P_b & \text{if } v \mid N_- \\ \varepsilon_v \mathrm{rec}_K(k_v^{-1}) P_b & \text{if } v \mid N_+ \\ P_b & \text{if } v \nmid N \end{cases}$$

Proof. (of the lemma)

For each v inert in K/F we have

$$\begin{aligned} \mathrm{inv}_v(B) = 1 &\iff B_v \simeq M_2(F_v) \\ &\iff \mathfrak{b} \in \mathrm{N}_{K_v/F_v}(K_v^\times) = \mathcal{O}_{F_v}^\times F_v^{\times 2} \\ &\iff 2 \mid \mathrm{ord}_v(\mathfrak{b}) \end{aligned}$$

As $\overline{j} = -j$, we have $\mathrm{nr}(j) = -j^2 = -\mathfrak{b}$ and

$$\mathrm{inv}_v(B) = 1 \iff 2 \mid \mathrm{ord}_v(\mathrm{nr}(j_v)).$$

If $v \mid N_-$. then $H_v = \mathcal{O}_{B_v}^\times$, where $\mathcal{O}_{B_v}$ is the unique maximal order in B_v hence $H_v \lhd B_v^\times$ and $B_v^\times / H_v^\times \simeq \mathbf{Z}$ by chosing some uniformizer. As H_v is normal in $B_v^\times$, the map

$$[\cdot j_v] : \mathrm{Sh}_H(G/Z, X)(\mathbf{C}) \longrightarrow \mathrm{Sh}_{j_v^{-1}Hj_v}(G/Z, X)(\mathbf{C})$$

is well-defined on $\mathrm{Sh}_H(G/Z, X)(\mathbf{C})$. Thus $[\mathscr{T}^\circ, bj_v]_{H\hat{F}^\times} = [\cdot j_v][\mathscr{T}^\circ, b]_{H\hat{F}^\times}$ and

$$\int_{\Delta_{bj_v}^\circ} \omega_\varphi = \int_{\Delta_b^\circ} [\cdot j_v]^* \omega_\varphi = \int_{\Delta_b^\circ} \pi_v(j_v)\omega_\varphi.$$

Decompose $\pi = \pi(\varphi) = \otimes_v' \pi_v$. We have

$$\pi_v : B_v^\times \xrightarrow{\mathrm{nr}} F_v^\times \xrightarrow{\mathrm{ord}_v} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} \{\pm 1\}.$$

Let us denote by α the following unramified character

$$\alpha : F_v^\times \xrightarrow{\mathrm{ord}_v} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} \{\pm 1\}$$

satisfying $\pi_v = \alpha \circ \mathrm{nr}$.
As $v \mid N_-$, E has multiplicative reduction in v. The character α is trivial if and only if E has split multiplicative reduction in v, i.e. $\varepsilon_v = -1$.

Hence
$$[\cdot j_v]^*\omega_\varphi = \alpha(\mathrm{nr}(j_v))\omega_\varphi = \begin{cases} \omega_\varphi & \text{if } \alpha = 1 \\ (-1)^{\mathrm{ord}_v(\mathrm{nr}(j))}\omega_\varphi & \text{otherwise.} \end{cases}$$
As $v \mid N_-$, $v \in \mathrm{Ram}(B)$ is inert in K/F and $\mathrm{inv}_v(B) = -1$, thus $2 \nmid \mathrm{ord}_v(\mathrm{nr}(j))$. Hence
$$[\cdot j_v]^*\omega_\varphi = \alpha(\mathrm{nr}(j_v))\omega_\varphi = \begin{cases} \omega_\varphi = -\varepsilon_v\omega_\varphi & \text{if } \alpha = 1 \\ -\omega_\varphi = -\varepsilon_v\omega_\varphi & \text{otherwise} \end{cases}$$
and $P_{bj_v} = -\varepsilon_v P_b$.
If $v \mid N_+$. then we fix some uniformizer ϖ_v of F_v and an isomorphism $B_v \simeq \mathrm{M}_2(F_v)$ which identifies K_v with the set of diagonal matrices and R_v with $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_{F,v}) \;\middle|\; \varpi_v^{\mathrm{ord}_v(N_+)} \mid c\right\}$.

As $\mathrm{inv}_v(B_v) = 1$, j_v is a local norm. There exists $k_v \in K_v$ such that $j_v = \mathrm{N}_{K_v/F_v}(k_v)$. We may assume that $j_v^2 = 1$. Moreover j_v is in the normalizer of $K_v^\times$ in $B_v^\times$ we thus identify j_v to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Set
$$W_v = \begin{pmatrix} 0 & 1 \\ \varpi_v^{\mathrm{ord}_v(N_+)} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{\mathrm{ord}_v(N_+)} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = k_v j_v.$$
This matrix is in the normalizer of R_v in B_v. As W_v normalize H_v,
$$[\mathscr{T}^\circ, bj_v]_{H\hat{F}^\times} = [\mathscr{T}^\circ, bk_v^{-1}W_v]_{H\hat{F}^\times} = [\cdot W_v][\mathscr{T}^\circ, bk_v^{-1}]_{H\hat{F}^\times}.$$
Decompose $\omega_\varphi = \bigotimes_{v|N_+}\omega_v \otimes \omega'$, where ω_v satisfies$[\cdot W_v]^*\omega_v = \varepsilon_v\omega_v$; then
$$\int_{\Delta^\circ_{bj_v}} \omega_\varphi = \varepsilon_v \int_{\Delta^\circ_{bk_v^{-1}}} \omega_\varphi.$$
As $b_v = 1$,
$$P_{bj_v} = \varepsilon_v \mathrm{rec}_K(k_v^{-1})P_b.$$
If $v \nmid N$. then by a similar calculation we obtain
$$P_{bj_v} = \mathrm{rec}_K(k_v^{-1})P_b.$$
□

End of the proof of Proposition 6.1.1. Lemma 6.1.2 implies that
$$c(P_1) = (-1)^{r-1}\prod_{v|N_-}(-\varepsilon_v)\prod_{v|N_+}\varepsilon_v\mathrm{rec}_K(k_v^{-1})P_1$$
and
$$\forall a \in K_{\mathbf{A}}^\times \qquad c(\mathrm{rec}_K(a)P_1) = (-1)^{r-1}\prod_{v|N_-}(-\varepsilon_v)\prod_{v|N_+}\varepsilon_v\mathrm{rec}_K(k_v^{-1})\mathrm{rec}_K(a)P_1.$$
As $P \in E(K)$, we know that $\mathrm{rec}_K(k^{-1})P = P$. Thus
$$c(P) = (-1)^{r-1}\prod_{v|N_-}(-\varepsilon_v)\prod_{v|N_+}\varepsilon_v P = (-1)^{r-1}(-1)^{|\{v|N_-\}|}\prod_{v\nmid\infty}\varepsilon_v P. \tag{3}$$

We have to show that $(-1)^{r-1}\prod_{v|N_-}(-\varepsilon_v)\prod_{v|N_+}\varepsilon_v = -\varepsilon$. For each $v \mid \infty$ we have $\varepsilon_v = -1$. Since $\prod_{v|\infty} = (-1)^d$, the sign in equation (3) is
$$(-1)^d\underbrace{\prod_v \varepsilon_v}_{=\varepsilon}(-1)^{r-1}(-1)^{|\{v|N_-\}|}.$$

Recall that $\{v \mid N_-\} = \mathrm{Ram}(B) \cap S_f$. As $|\mathrm{Ram}(B)|$ is even, we have
$$(-1)^{|\{v|N_-\}|} = (-1)^{|\mathrm{Ram}(B)\cap S_\infty|} = (-1)^{d-r}.$$
Hence
$$c(P) = (-1)^d\varepsilon(-1)^{r-1}(-1)^{|\{v|N_-\}|}P = -\varepsilon P.$$

□

Remark 6.1.3. The above computations are a particular case of a result of Prasad, [Pra96] Theorem 4, which asserts that if $\mathrm{Hom}_{K_v^\times}(\pi_v, \mathbf{1}) \neq \{0\}$, then the non trivial element in $\mathrm{N}_{B_v^\times}(K_v^\times)\backslash K_v^\times$ acts on $\mathrm{Hom}_{K_v^\times}(\pi_v, \mathbf{1})$ by multiplication by $\mathrm{inv}_v(B)\varepsilon_v = \mathrm{inv}_v(B)\varepsilon(\pi_v, \frac{1}{2}) \in \{\pm 1\}$.

6.2. **Orthogonal Shimura manifolds.** Until the end of this paper we shall assume $h_F^+ = 1$.

Let us recall some definitions used by Kudla [Kud97] in the particular case $r = 1$. Let $n \in \mathbf{Z}_{\geq 0}$ and let (V, Q) be a quadratic space over F of dimension $n+2$. We assume that the signature of $V \otimes_{F,\tau_j} \mathbf{R}$ is

$$(n, 2) \times (n+1, 1)^{r-1} \times (n+2, 0)^{d-r}.$$

Denote by D the symmetric space of $G = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GSpin}(V)$. D is the product of the oriented symmetric spaces of $V_j = V \otimes_{\tau_j, F} \mathbf{R}$. Thus $D = D_1 \times \dots D_d$, where D_j is the set of oriented positive subspaces in V_j of maximal dimension. For each $x \in V$ let x_j be the image of x in V_j. Assume that $Q(x)$ is totally positive. Set $V_x = x^\perp$, $G_x = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GSpin}(V_x)$ and for each $j \in \{1, \dots, d\}$

$$D_{x_j} = \{z \in D_j \ z \perp x_j\}.$$

We shall focus on the following real cycle on the Shimura manifold $G(\mathbf{Q})\backslash D \times G(\widehat{\mathbf{Q}})/H$.

Definition 6.2.1. Let H be an open compact subgroup in $G(\widehat{\mathbf{Q}})$ and $g \in G(\widehat{\mathbf{Q}})$. The cycle $Z(x, g; H)$ is defined to be the image of the map

$$Z(x, g; H) : \left\{ \begin{array}{ccc} G_x(\mathbf{Q})\backslash D_x \times G_x(\widehat{\mathbf{Q}})/H_x^g & \longrightarrow & G(\mathbf{Q})\backslash D \times G(\widehat{\mathbf{Q}})/H \\ G_x(\mathbf{Q})(y, u)H_x^g & \longmapsto & G(\mathbf{Q})(y, ug)H\widehat{F}^\times, \end{array} \right.$$

where H_x^g denotes $G_x(\widehat{\mathbf{Q}}) \cap gHg^{-1}$.

Example (including Proposition 6.2.2) : Fix $D_0 \in F$ satisfying

$$\left\{ \begin{array}{ll} \tau_j(D_0) > 0 & \text{if } j \in \{1, r+1, \dots, d\} \\ \tau_j(D_0) < 0 & \text{if } j \in \{2, \dots, r\} \end{array} \right.$$

Set

$$(V, Q) = (B^{\mathrm{Tr}=0}, D_0 \cdot \mathrm{nr}).$$

$(V \otimes_{F,\tau_j} \mathbf{R}, \tau_j \circ D_0 \cdot \mathrm{nr})$ has signature

$$\left\{ \begin{array}{ll} (1, 2) & \text{if } j = 1 \\ (2, 1) & \text{if } j \in \{2, \dots, r\} \\ (3, 0) & \text{if } j \in \{r+1, \dots, d\}. \end{array} \right.$$

Let $G = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GSpin}(V)$. The action of $B^\times$ on V by conjugation induces an isomorphism

$$\begin{array}{ccc} B^\times & \xrightarrow{\sim} & \mathrm{GSpin}(V) \\ b & \longmapsto & (v \mapsto bvb^{-1}), \end{array}$$

thus $G \simeq \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$.

Let $x \in V$ such that $Q(x) \gg 0$, and denote by x_j its image in $V \otimes_{F,\tau_j} \mathbf{R}$. Denote by K the quadratic extension $F + Fx$ and $T = \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)/\mathrm{Res}_{F/\mathbf{Q}}(\mathbf{G}_m)$ as above. Let q be the inclusion $K \hookrightarrow V \to B$.

Proposition 6.2.2. *The set*

$$D_x = D_{x_1} \times \dots \times D_{x_r}$$

is a $q(T(\mathbf{R}))^\circ$-orbit in D whose projection on D_1 is a point.

Proof. As $x \in V$, $\mathrm{Tr}(x) = 0$ and $x^2 = -\mathrm{nr}(x) = -\frac{Q(x)}{D_0} \in F^\times$. Let $j \in \{1, \dots, r\}$. We have $\tau_j(Q(x)) > 0$ hence $\tau_j(x^2)\tau_j(D_0) < 0$. Thus τ_1 ramifies in K and $\tau_2, \dots, \tau_r$ are split. Moreover $q_1(K^\times)$ fixes x_1 by defintion of K.

□

Let us focus on the general case when V has dimension n. Fix $t \in F$ satisfying $\forall j \in \{1, \ldots, r\}\ \tau_j(t) > 0$. $G(\widehat{\mathbf{Q}})$ acts on $\Omega_t = \{x \in V(F) \mid \quad Q(x) = t\}$ by conjugation.

Let φ be a Schwartz function on $V(\widehat{F})$. Assume $\Omega_t \neq \emptyset$ and fix $x \in \Omega_t$. Denote by $Z(y, \varphi; H)$ the following sum

$$Z(t, \varphi; H) = \sum_{g \in G_x(\widehat{\mathbf{Q}}) \backslash G(\widehat{\mathbf{Q}})/H\widehat{F}^\times} \varphi(g^{-1} \cdot x) Z(x, g; H).$$

Proposition 4.3.1 showed that for $n = 1$ $[Z(x, g; H)] = 0 \in H_{r-1}(\mathbf{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{C})$. A natural invariant to consider is the refined class

$$\{Z(t, \varphi; H)\} = \omega \mapsto J_b^\beta \in \frac{(\mathrm{Harm}^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}))^*}{\mathrm{Im}(H_r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{Z}) \to \mathrm{Harm}^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}))^*)},$$

where $\mathrm{Harm}^r(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}))$ is the set of harmonic differential forms on $\mathrm{Sh}_H(G/Z, X)(\mathbf{C})$.

In order to adapt the work of Yuan, Zhang and Zhang, we need the following conjecture

Conjecture 6.2.3. *In the situation of the above example* $(V, Q) = (B^{\mathrm{Tr}=0}, D_0 \cdot \mathrm{nr})$*, the sum*

$$\sum_{\substack{t \in \mathcal{O}_F \\ t \gg 0}} \{Z(t, \varphi; H)\} q^t$$

is a Hilbert modular form of weight 3/2.

In [YZZ09], the authors work by induction. To apply their method we would need to prove that the refined classes $\{Z(t, \varphi; H)\}$ are compatible with the tower of varieties attached to quadratic spaces $V_x \hookrightarrow V$ of signature $(n, 2) \times (n+1, 1)^{r-1} \times (n+2, 0)^{d-r}$ (in which case a generalization of [KM90] should imply that $\sum_{\substack{t \in \mathcal{O}_F \\ t \gg 0}} [Z(t, \varphi; H)] q^t$ is a Hilbert modular form of weight $\frac{n}{2} + 1$ with coefficients in $H^{r+1}(\mathrm{Sh}_H(G/Z, X)(\mathbf{C}), \mathbf{C})$).

6.3. **A Gross-Kohnen-Zagier-type conjecture.**

The Bruhat-Tits tree. In this section we recall some basic facts about the Bruhat-Tits tree (see [CJ] and [Vig80]).

Let v be a finite place of F. The vertices of the Bruhat-Tits tree of $\mathrm{PGL}_2(F_v)$ are the maximal orders of $\mathrm{M}_2(F_v)$. Such maximal orders are endomorphism rings of lattices in F_v^2 ([Vig80], lemme 2.1). There is an oriented edge between two vertices $\mathcal{O}_1$ and $\mathcal{O}_2$ if and only if there exist L_1, L_2 lattices in F_v^2 such that $\mathcal{O}_i = \mathrm{End}(L_i)$, $L_2 \subset L_1$ and $L_1/L_2 \simeq \mathcal{O}_{F_v}/\varpi_v \mathcal{O}_{F_v}$. The intersection of the source and the target of paths of length n correspond to level v^n Eichler orders.

Fix some quadratic extension K/F. This data allow us to organize the Bruhat-Tits tree. Let $\Psi : K_v \hookrightarrow \mathrm{M}_2(F_v)$ be a F_v-embedding of K_v. Let $\mathrm{M}_0(N)$ be the set of matrices in $\mathrm{M}_2(F_v)$ which are upper triangular modulo N. If

$$\Psi(\mathcal{O}_{K_v}) = \Psi(K_v) \cap \mathrm{M}_0(N),$$

we say that Ψ has level N. We can organize the vertices of the tree in "levels", by privileging a direction. Each level corresponds to a level of embedding relatively to $\mathcal{O}_{K_v}$ i.e. to orders which are in the same orbit under $K_v^\times$. The maximal orders in $\mathrm{PGL}_2(F_v)$ which are maximally embedded are on the bottom of the tree.

Figures 2, 3 and 4 illustrate the dependence on the ramification type of v in K.

Darmon's points, Kudla's program and a Gross-Kohnen-Zagier-type theorem. Recall that $H = (R \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times$, where R is an Eichler order of B of level N_+ and that $K = F + Fx$ satisfies the following Heegner hypothesis.

Hypothesis 6.3.1. *Each prime* $\mathfrak{p} \mid N_+$ *splits in* K *and each prime* $\mathfrak{p} \mid N_-$ *is inert in* K.

The group G_x is isomorphic to $K^\times$ and $Z(x, 1; H)$ is the image of $K^\times \backslash D_x \times \widehat{K}^\times / H$ in $\mathrm{Sh}_H(G, X)(\mathbf{C})$. Note that

$$Z(x, 1; H) = \mathscr{T}_1^1 + t_1(\mathscr{T}_1^1),$$

where $\mathscr{T}_1^1 = [\cup_{u \in \pi_0(T(\mathbf{R}))} q(u) \cdot \mathscr{T}^\circ, 1]_{H\widehat{F}^\times}$.

Let $\varphi = \mathbf{1}_{\widehat{R}^{\mathrm{Tr}=0}}$. We are able to prove an analogue of Proposition A.I.1 of [Kud04] when $N = 1$, $B = \mathrm{M}_2(F)$, $R = \mathrm{M}_2(\mathcal{O}_F)$, $t = Q(x) = D_0 \mathrm{nr}(x) \in F$ and $K = F + Fx$ is such that $K \cap R = \mathcal{O}_K$ and $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$. Set $c_1(\mathscr{T}_1^1) = \{[t_1(x), b]_{H\widehat{F}^\times},\ b \in \widehat{B}^\times\}$.

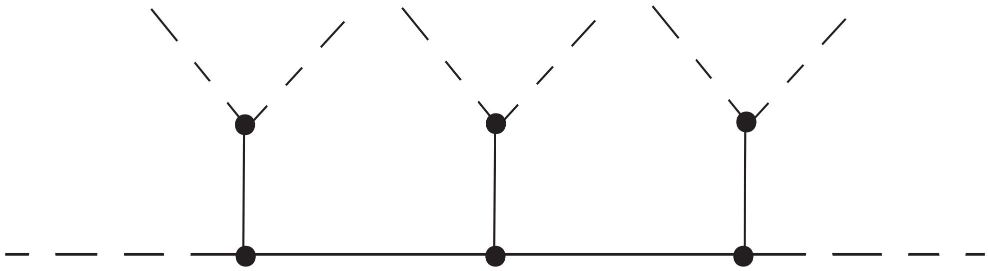


FIGURE 2. Bruhat-Tits tree of $\mathrm{PGL}_2(F_v)$ when v is split.

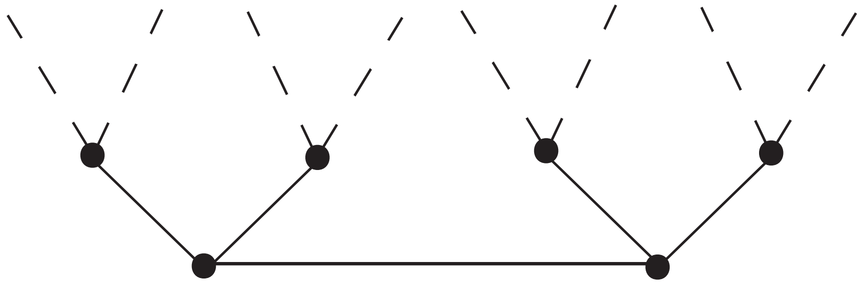


FIGURE 3. Bruhat-Tits tree $\mathrm{PGL}_2(F_v)$ when v is ramified.

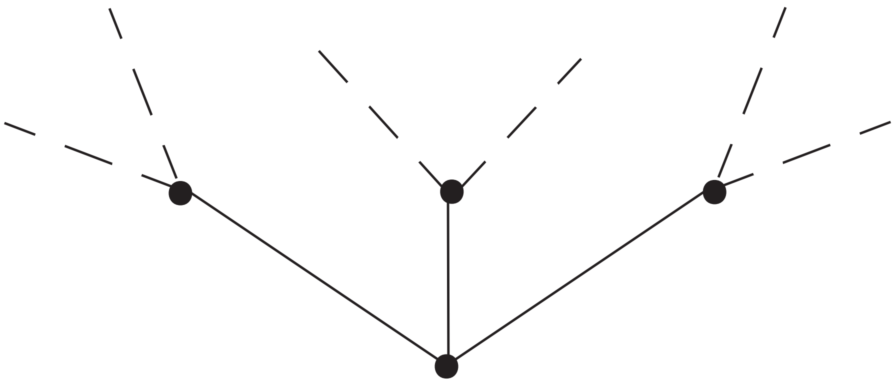


FIGURE 4. Bruhat-Tits tree of $\mathrm{PGL}_2(F_v)$ when v is inert.

Proposition 6.3.2. *If $N = 1$, $r = d$, $B = \mathrm{M}_2(F)$, $H = \widehat{R}^\times$ with $R = \mathrm{M}_2(\mathcal{O}_F)$ and if $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$, then $Z(t, \varphi; H)$ is equal to*

$$Z(x, 1; H) = \mathscr{T}_1^1 + c_1(\mathscr{T}_1^1) = \mathscr{T}_1^1 - \varepsilon \mathscr{T}_1^1.$$

Remark 6.3.3. Under the strong hypotheses above, $\varepsilon = (-1)^d$ and the cycle obtained is zero when d is even.

Proof. By definition

$$Z(t, \varphi; H) = \sum_{g \in \widehat{K}^\times \backslash \widehat{B}^\times / \widehat{R}^\times} \mathbf{1}_{\widehat{R}^{\mathrm{Tr}=0}}(g^{-1} \cdot x) Z(x, g; H).$$

We have to determine $g \in \widehat{K}^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ satisfying $g^{-1} x g \in \widehat{R}^{\mathrm{Tr}=0}$, i.e. $x \in g \widehat{R}^{\mathrm{Tr}=0} g^{-1}$. As $F^\times \subset K^\times$,

$$\widehat{K}^\times \backslash \widehat{B}^\times / \widehat{F}^\times \widehat{R}^\times = \prod_v{}' K_v^\times \backslash B_v^\times / R_v^\times = \prod_v{}' K_v^\times \backslash B_v^\times / F_v^\times R_v^\times.$$

This allows us to work locally with $K_v^\times\backslash B_v^\times/F_v^\times R_v^\times$, which is identified to the $K_v^\times$-orbits of maximal orders of $\mathrm{PGL}_2(F_v)$. This gives the following condition, $x_v \in g_v R_v g_v^{-1}$.

First let us consider those $g_v \in B_v^\times/R_v^\times F_v^\times$ satisfying $x_v \in g_v R_v g_v^{-1}$. The ring $g_v R_v g_v^{-1}$ is a maximal order containing x_v. Using the fact that $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$, we have

$$x_v \in g_v R_v g_v^{-1} \iff g_v R_v g_v^{-1} \cap K_v = \mathcal{O}_{K_v}.$$

Hence the maximal order $g_v R_v g_v^{-1}$ is maximally embedded in K_v. It is identified to a vertex at the lowest level of the Bruhat-Tits tree. As each vertex at the same level is in the same $K_v^\times$-orbit, we have

$$\forall v \qquad g_v = 1 \in K_v^\times\backslash B_v^\times/F_v^\times R_v^\times.$$

Thus $Z(t,\varphi;H) = Z(x,1;H)$ and as D_{x_1} is a set of two points, $Z(x,1;H)$ is identified with $\mathscr{T}_1^1 + c_1(\mathscr{T}_1^1) = \mathscr{T}_1^1 - \varepsilon\mathscr{T}_1^1$, thanks to Proposition 6.1.1.

□

We now consider the case when $N = N_+N_- \neq 1$ is prime to $d_{K/F}$. The following proposition is true even if $B \neq \mathrm{M}_2(F)$ but we still assume that R is an Eichler order of level N_+ and $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$.

Proposition 6.3.4. *Let N be the conductor of E. If N is prime to $d_{K/F}$, then*

$$Z(t,\varphi;H) = \prod_{v|N}(1 + \mathrm{inv}_v(B)\varepsilon_v)Z(x,1;H).$$

Proof. The proof is analogous to the proof of Proposition 6.3.2. Let us first compute the number of terms in $Z(t,\varphi;H)$. We need to determine for each v the number of $K_v^\times$-orbits of oriented paths of length $\mathrm{ord}_v(N_+)$ in the Bruhat-Tits tree; this is equal to the number of g_v such that $x_v \in g_v R_v g_v^{-1}$.

- If $v \nmid N$ then the same argument as in Proposition 6.3.2 shows that there is only one orbit.
- If $v \mid N_-$, B_v is ramified and v is inert in K. Hence $K_v^\times\backslash B_v^\times/R_v^\times F_v^\times = \{1, \pi_v\}$ where $\pi_v \in B_v^\times$ is an element whose reduced norm has order 1 at v; π_v corresponds to the Atkin-Lehner involution.
- If $v \mid N_+$, v splits in K. Denote by v^δ the level of the order R_v. Each Eichler order of level v^δ is the intersection of the origin and the target of an oriented path of length δ. By hypothesis those orders are maximally embedded in K_v and the path corresponding to $g_v R_v g_v^{-1}$ is contained in the lowest level of the tree. As $K_v^\times$ acts by translations on this level, there are exactly two $K_v^\times$-orbits corresponding to g_v depending on the orientation. We have g_v^+ and g_v^- which are exchanged by the Atkin-Lehner involution corresponding to $\begin{pmatrix} 0 & \varpi_v \\ 1 & 0 \end{pmatrix}$.

Let n be the number of prime ideals in the decomposition of N. The sum $Z(t,\varphi;H)$ has 2^n factors. Let W be the sets of these factors. By definition $Z(x,g;H) = [\cdot g]Z(x,1;H)$. Using Proposition 6.1.1 we obtain

$$Z(t,\varphi;H) = \sum_{g\in W}[\cdot g]Z(x,1;H) = \prod_{v|N}(1 + \mathrm{inv}_v(B)\varepsilon_v)Z(x,1;H).$$

□

Let us conclude this paper by another conjecture. Assume that $E(F)$ has rank 1. Denote by P_0 some generator of $E(F)$ modulo torsion. For each $t \in \mathcal{O}_F$ totally positive such that (t) is square free and prime to $d_{K/F}$, denote by $K[t]$ the quadratic extension

$$K[t] = F(\sqrt{-D_0 t}),$$

which satisfies the hypothesis used to build Darmon's points. Let $P_{t,1}$ be Darmon's point obtained for $K[t]$ and $b = 1$, and set

$$P_t = \mathrm{Tr}_{K[t]_1^+/F} P_{t,1}.$$

The point P_t lies in $E(F)$ and there exists an integer $[P_t] \in \mathbf{Z}$ such that

$$P_t = [P_t]P_0 \text{ modulo torsion}.$$

Proposition 6.3.4 together with Conjecture 6.2.3 suggest the following (as in Conjecture 5.3 of [DT08]).

Conjecture 6.3.5. *There exists some Hilbert modular form g of level 3/2 such that the $[P_t]$s are proportional to some Fourier coefficients of g.*

Remark 6.3.6. Using the analogy with the Gross-Kohnen-Zagier theorem, the integers $[P_t]$ should be (proportional to) square roots of $L(E_{-D_0 t}, 1)$, where $E_{-D_0 t}$ is the twist of E by $-D_0 t$.

Let us end this paper with two open questions.

Question 6.3.7. Does Bruinier's genralization of Borcherds products [Bru] give anything interesting in this situation ?

It is natural to expect that results of Cornut and Vatsal [CV07, CV05] hold also for Darmon's points.

Question 6.3.8. Would it be possible to deduce such a result from suitable equidistribution properties for the real tori $\mathscr{T}_b^\circ$?

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